Among the most popular techniques for portfolio insurance strategies that are used nowadays, the so-called “Constant Proportion Portfolio Insurance” (CPPI) allocation simply consists in reallocating the risky part of a portfolio according to the market conditions. This general method crucially depends upon a parameter - called the multiple - guaranteeing a predetermined floor whatever the plausible market evolutions. However, the unconditional multiple is defined once and for all in the traditional CPPI setting; we propose in this article an alternative to the standard CPPI method, based on the determination of a conditional multiple. In this time-varying framework, the multiple is conditionally determined in order the risk exposure to remain constant, but depending on market conditions. We thus propose to define the multiplier as a function of an Extended Expected Value-at-Risk.

After briefly recalling the portfolio insurance principles, the CPPI framework and the main properties of the conditional or unconditional multiples, we present and compare several models for the conditional multiple and estimate them using parametric, semi-parametric and non-parametric methods. We then compare the conditional multiple models when it is used in insured portfolio strategies and introduce an original time-varying strategy - called Time-varying Proportion Portfolio Insurance (TPPI) - whose aim is to adapt the current exposition to market conditions following a traditional risk management philosophy. Finally, we use an option valuation approach for measuring the gap risk in both conditional and unconditional approaches.

Key Words: CPPI, Portfolio Insurance, VaR, CAViaR, Quantile Regression, Dynamic Quantile Model, Expected Shortfall, Extreme Value.

JEL Classification: G11, C13, C14, C22, C32.
« A Time-varying Proportion Portfolio Insurance Strategy 
based on a CAViaR Approach »

Preliminary Draft
- March 2008 -

Abstract

Among the most popular techniques for portfolio insurance strategies in use nowadays, the so-called “Constant Proportion Portfolio Insurance” (CPPI) allocation simply consists in reallocating the risky part of a portfolio according to the market conditions. This general method crucially depends upon a parameter - called the multiple - guaranteeing a predetermined floor whatever the plausible market evolutions. However, the unconditional multiple is defined once and for all in the traditional CPPI setting; we propose in this article an alternative to the standard CPPI method, based on the determination of a conditional multiple. In this time-varying framework, the multiple is conditionally determined in order the risk exposure to remain constant, but depending on market conditions. We thus propose to define the multiplier as a function of an Extended Expected Value-at-Risk.

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1. Introduction

Leland and Rubinstein (1976) first show that an optional asymmetric performance structure can be reached using some portfolio insurance strategies. Thanks to dynamic allocation strategies, insured portfolios are protected against large falls by a contractually guaranteed predetermined floor and they take partially advantage of market performances. A portfolio insurance trading strategy is defined to guarantee a minimum level of wealth at a specified time horizon, and to participate in the potential gains of a reference portfolio (Grossman and Villa, 1989; Basak, 2002). Thus the investor reduces his downside risk and participates to market rallies. The most prominent examples of dynamic versions are the Constant Proportion Portfolio Insurance (CPPI) strategies and Option–Based Portfolio Insurance (OBPI) strategies with synthetic puts.\footnote{Option-Based Portfolio Insurance (OBPI) with synthetic puts is introduced in Leland and Rubinstein (1976). Synthetic is understood here in the sense of a trading strategy in traded assets that create the put. In a complete financial market model, a perfect hedge exists (i.e. a self-financing and duplicating strategy). In contrast, the introduction of market incompleteness impedes the concept of perfect hedging.}

On a micro-economic perspective, such strategies using insurance properties are thus rationally preferred by individuals that are specially concerned by extreme losses and completely risk averse for values below the guarantee (or floor). We propose in this paper a new applied financial strategy that helps the investor in realizing his objectives for most of the market conditions.

The Constant Proportion Portfolio Insurance (CPPI) is introduced by Perold (1986) on fixed income assets. Black and Jones (1987) extend this method by using equity based underlying assets. In that case, the CPPI is invested in various proportions in a risky asset and in a non-risky one to keep the risk exposure constant. CPPI strategies are very popular: they are commonly used in hedge funds, retail products or life-insurance products. The main difficulty of the CPPI
strategy is to determine the parameter defining the portfolio risk exposure, known as the multiple. Banks directly bear the risk of the guaranteed portfolios they sell: at maturity, if the guaranteed floor is not reached, the banks have to compensate the loss with their own capital. The sharpest determination of the multiple is the main actual challenge of the CPPI strategy. Unconditional multiple determination methods have been developed in the literature such as the extreme value approach to the CPPI method (see Bertrand and Prigent, 2002 and 2005; Prigent and Tahar, 2005). But all these unconditional setting methods of the multiple reduce the risk exposure to the risky asset exposure. Thus these traditional unconditional methods do not take into account that the risk of the risky underlying asset changes according, for instance, to market conditions. We develop here a setting method of conditional multiple to the CPPI strategy to keep a constant risk exposure.

The aim of this paper is to model and compare different ways for estimating the multiple, conditionally to market evolutions, when keeping a constant risk exposure defined by the Value-at-Risk (VaR) or by the Expected Shortfall. The conditional multiple is estimated using parametric, non-parametric and semi-parametric methods. Several dynamic quantile models are used as extensions of CAViaR models (see Engle and Manganelli, 2004; Gouriéroux and Jasiak, 2006) based on quantile regression estimations (see Koenker and Basset, 1978; Mukherjee, 1999).

After having briefly recalled portfolio insurance principles, the CPPI method and properties that conditional or unconditional multiple have to follow (section 2), we justify, describe and estimate the conditional multiple model when keeping a constant exposition to the risk using parametric, semi-parametric and non-parametric methods (section 3 and 4). We illustrate then the Time-varying Proportion Portfolio Insurance in section 5. The introduction of market incompleteness and model risk may impede the concept of dynamic portfolio insurance. Measuring the risk that the value of a CPPI strategy is less than the floor (or guaranteed amount) is therefore of practical importance. For example, the introduction of trading restrictions (liquidity problem…) is one justification to take into account the gap risk in the sense that a CPPI
strategy cannot be adjusted adequately. Thus, an additional option is often written. The option is exercised if the value of the CPPI strategy is below the floor. Using this hedging approach, we finally evaluate the gap risk between conditional and unconditional approaches in a proportion portfolio insurance framework according to different option valuation model (section 6).

2. Cushioned Portfolio Evidences

Portfolio insurance is defined to allow investors to recover, at maturity, a given proportion of their initial capital. One of the standard portfolio insurance methods is the Constant Proportion Portfolio Insurance (CPPI). This strategy is based on a specific dynamic allocation on a risky asset and on a riskless one to guarantee a predetermined floor. The properties of CPPI strategies are studied extensively in the literature, (Bookstaber and Langsam, 2000; Black and Perold, 1992). A comparison of OBPI and CPPI is given in Bertrand and Prigent (2005). The literature also deals with the effects of jump processes, stochastic volatility models and extreme value approaches on the CPPI method (Bertrand and Prigent, 2002 and 2003).

The management of cushioned portfolio follows a dynamic strategic portfolio allocation. The floor, denoted \( F_t \), is the minimum value of the portfolio, which is acceptable for an investor at maturity. The value of the covered portfolio, denoted \( V_t \), is invested in a risky asset denoted by \( S \) and a non-risky asset denoted by \( B \). The proportion invested in the risky asset varies relatively to the amount invested in the non-risky asset, in order to insure at any time the guaranteed floor. Hence, even if the market is downward sloping at the investment horizon \( T \), the portfolio will keep at maturity a value equal to the floor, (i.e. a predetermined percentage of the capital deposit at the beginning of the management period). At maturity, the theoretical guaranteed value cannot be obviously higher than the value initially invested and capitalized at the non-risky rate \( r^B \), denoted by \( V_0 e^{rT} \).
The cushion, denoted by $c_t$, is defined as the spread (which can vary across time) between the portfolio value and the value of the guaranteed floor. Therefore, it satisfies:

$$c_t = V_t - F_t$$  \hspace{1cm} (1)

Thus, the cushion is the maximal theoretical amount, which we can lose over a period without reducing the guaranteed capital. The ratio between the risky asset and the cushion corresponds, at each time, to the target multiple, denoted by $m_t$. The multiple reflects the maximal exposure of the portfolio. The cushioned management strategy aims at keep a constant proportion of risk exposure. The position in the risky asset, denoted $e_t$, has to be proportional to the cushion. Thus, we have at any time:

$$e_t = m_t \times c_t$$  \hspace{1cm} (2)

It means that the amount invested in the risky asset is determined by multiplying the cushion by the multiple.

At any time, the fluctuating multiple moves away from its target value. This is the reason why a third parameter is introduced, the tolerance, denoted by $\tau$, to determine when the portfolio has to be rebalanced. If after the fluctuation of the risky asset, the remaining multiple, denoted by $m_t^*$, moves away from its target value of a superior percentage of the tolerance, there will exist adjustments (thus transaction fees).

Therefore, we have:

$$\forall t \in [0, \ldots, T], \quad m_t^* \in [m_t \times (1-\tau), m_t \times (1+\tau)]$$  \hspace{1cm} (3)

The problem of the cushion management is the determination of the target multiple. For instance, if the risky asset drops, the value of the cushion must remain (by definition) superior or equal to zero. Therefore, the portfolio based on the cushion method will have (theoretically) a value superior or equal to the floor. However, in case of a drop of the risky underlying asset, the higher the multiple, the higher the cushion and the exposure tends rapidly to zero. Nevertheless,
before the manager can rebalance his portfolio, the cushion allows absorbing a shock inferior or equal to the inverse of the superior limit of the multiple.

3. Quantile Models for Cushioned Portfolios

The main parameter of a CPPI dynamic strategy is the multiple. The first purpose of this paper is to determine conditional multiple models and to study them using several estimation methods.

The first model is based on Value-at-Risk (VaR). Nowadays, VaR is considered as the standard measure to quantify market risk. Thus, it seems appropriate to use it for modelling the conditional multiple, which allows the hedged portfolio to keep a constant exposure to risk.

In this section, the intuition of using VaR to model the conditional multiple has to be justified. This first conditional multiple model based on VaR is then detailed, before reviewing the main quantile models and estimation methods used in this paper.

The optimality of an investment strategy depends on the risk profile of the investor. In order to determine the optimal rule, one has to decide what strategy to adopt according to the expected utility criterion. Thus, portfolio insurers can be modelled by utility maximizers where the optimization problem is given under the additional constraint that the value of the strategy is above a specified wealth level. In a complete market, the CPPI can be characterized as expected utility maximizing when the utility function is piecewise HARA and the guaranteed level is growing with the riskless interest rate (Kingston, 1989). This argument can be no more valid if additional frictions are introduced such as for example trading restrictions. Without postulating completeness, we refer to the works of Cox and Huang (1989), Brennan and Schwartz (1989), Grossman and Villa (1989), Grossman and Zhou (1993, 1996), Basak (1995), Cvitanic and Karatzas (1995 and 1999), Browne (1999), Tepla (2000 and 2001) and El Karoui et al (2005). Mostly, the solution of the maximization problem is given by the unconstrained problem including a put option. Obviously, this is in the spirit of the OPBI method. But, the introduction
of various sources of market incompleteness in terms of stochastic volatility and trading restrictions makes the determination of an optimal investment rule under minimum wealth constraints quite difficult (if not impossible). For example, if the payoff of a put (or call) option is not attainable, the standard OBPI approach is no more viable since a dynamic option replication must be introduced. It explains why the CPPI method has become so popular among practitioners. In incomplete markets, hedging strategies depend on some dynamic risk measure (Schweizer, 2001). In this framework, the use of the return quantile inverse to determine the conditional multiple is justified by its precise definition of the expected maximum loss. The centile approach can also directly be interpreted as a VaR, which is considered nowadays as a standard measure to quantify market risk. Moreover, the VaR is now directly used in profit and loss paradigm. Thus, in portfolio selection model under shortfall constraints introduced in the work by Roy (1952) on safety-first theory and developed by Lucas and Klaassen (1998), the shortfall constraint is defined such that the probability of the portfolio value falling below a specified disaster level is limited to a specified disaster probability. Campbell et al. (2001) have used the VaR to define the shortfall constraint in order to develop a market equilibrium model for portfolio selection. Following this work, the risky asset exposition is driven by a conditional multiple determined by the inverse of a shortfall constraint quantified by the VaR. The hedging depends in fact on this risk measure. The conditional multiple allows the hedged portfolio to keep a constant exposure to risk defined by the VaR.

To be protected in a CPPI setting, the multiple of the insured portfolio has to stay smaller than the inverse of the underlying asset maximum drawdown, until the portfolio manager can rebalance his position. Additionally, to obtain a convex cash flow with respect to the risky asset

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2 The market imperfection can be caused by trading restrictions. The price process of the risky asset, is driven by a continuous-time process, while trading is restricted to discrete time. Therefore, the effectiveness of the OBPI approach is given by the effectiveness of a discrete-time option hedge. The error of time-discretizing a continuous-time hedging strategy for a put (or call) is extensively studied in the literature (cf. Boyle and Emanuel, 1980; Russel and Schachter, 1994; Bertsimas et al., 1998; Mahayni, 2003; Talay and Zheng, 2003; Hayashi and Mykland, 2005).
return, the investor has to require a multiple higher than one. Moreover, to keep a portfolio value superior to its floor, the cushion has to be positive. Thus, we have:

\[
\forall t \in [0,\ldots,T-1] \quad c_{t+1}/c_t \geq 0 \Rightarrow X_t \leq 1+(1-m_t)r_{t+1}^b/m_t
\]

and:

\[
c_{t+1}/c_t \geq 0 \Rightarrow X_t \leq (m_t)^{-1}
\]

where \( c_t \) is the cushion value at time \( t \), \( X_t \) is the opposite of the underlying asset rate of return \( r_t^S \) (\( X_t \leq -\Delta S_t / S_t \)), \( m_t \) is the multiple and \( r_t^b \) is the risk free rate. To be protected, a CPPI has to fulfil the following condition, at any given time \( t \):

\[
m_t \leq [\overline{X}_t]^{-1}
\]

(4)

with \( \overline{X}_t = \max \{ X_s \}_{0 \leq s \leq t} \), is the maximum of the realizations of the random variable for \( t \) between \( \theta \) and \( T \).

For a capital guarantee constraint at a significance level \( \alpha \%) \), the multiple must be lower than the inverse of the \( \alpha \) conditional-quantile of the asset return distribution, that is:

\[
P_t[ c_{t+1}/c_t \geq 0 ] \geq 1-\alpha \iff m_t (1+\tau) \leq -\{ Q_{\alpha,t+1} \}^{-1} \quad \forall t \in [0,\ldots,T]
\]

(5)

with:

\[
\left\{ Q_{\alpha,t+1} = \inf_{r_s \in \mathbb{R}} \{ r_{t+1} | F_{t+1}(r_{t+1}) \geq \alpha \} \right\}
\]

\[
m_t (1-\tau) \geq 1
\]

where \( T \) is the terminal date, \( Q_{\alpha,t+1} \) is the \( \alpha \) quantile of the conditional distribution denoted \( F_t(.) \) of the risky periodic asset returns and \( r_{t+1}^S \) is a future periodic return.

Thus, the target multiple can be interpreted as the inverse of the maximum loss that can bear the cushioned portfolio before the re-balancing of its risky component, at a given confidence level. For a given parametric model, explicit solutions for the upper bound on the multiple can be provided from Relation 5. (see Appendix 1 for an example when the risky asset

\( ^3 \) By definition, the multiple is strictly positive. With a multiple equal to 1, the protection is absolute: the risky asset exposition is then equal to the value of the cushion. A multiple inferior to one is therefore irrational. Nevertheless, even if the multiple was smaller than one, \((1-m_t)r_t^S\) is negligible relative to \( m_t \) and this property would be further verified.
price follows a general marked point process). However, without specific assumptions on the market dynamics, upper bounds on the multiple can also be determined from market conditions.

Our first aim is to compare several conditional estimation methods of the target multiple within the cushioned portfolio framework. We use a quantile hedging approach based on the conditional VaR. Within this framework, the multiple can be modelled by the inverse of the first percentile conditional on the distribution of the asset return augmented of the \( \theta \) quantile of the excess return comparatively at the centile, \( D_i(\theta) \). The target multiple can be written as:

\[
m_t = \left[ C_i(r_s^T; \beta) + D_i(\theta) \right]^{-1}
\]  

where \( C_i(r_s^T; \beta) \) is the first percentile of the conditional distribution of returns of the underlying asset \( r_s^T \), corresponding to the periodic return of the risky part of the portfolio covered, \( \beta \) is the vector of unknown parameters of the conditional percentile function, and \( D_i(\theta) \) represents the quantile of the return in excess observed in case of overtaking the conditional centile.

The maximum of potential drops in the risky asset value is estimated at each period adding at the first conditional percentile estimated, the quantile of the additional return observed in case of overtaking. This last term whose estimation denoted by \( \hat{D}_i(\theta) \) is calculated over the period by using the relation:

\[
\forall t \in [0, \ldots, T] \quad \hat{D}_i(\theta) \approx - \max_{t-n_{assid}} \left[ C_i(r_s^T; \beta) - r_s, 0 \right]
\]  

where \( C_i(r_s^T; \beta) \) is the centile of the risky asset return estimated by the conditional model considered above and \( n \) is the total number of observations before \( t \).

The multiple can be modelled by the inverse of the first conditional percentile of the risky asset return distribution augmented by the \( \theta \) quantile of the excess return relatively to this centile. The multiple is modelled as a function of centile models, which is the VaR at a 99% probability confidence level (denoted VaR\(_{99\%}\), hereafter). We present below the main VaR models and estimation method used in this paper for estimating this first model of multiple.
The financial regulation has generated a quiet impressive literature about Value-at-Risk (VaR) calculation, which has emerged to provide an accurate assessment of the exposure to market risk of portfolio. Nowadays, VaR can be considered as the standard approach to quantify market risk. Different approaches have been proposed for estimating conditional tail quantiles of financial returns. It measures the potential loss of a given portfolio over a prescribed holding period at a given confidence level. The VaR is a quantile function, so an inverse of the cumulative distribution function. The common approach to model dynamic quantile is to specify a conditional quantile at a risk level $\alpha$ as a function of conditioning variables known at a given time $t$:

$$f_t(\alpha) = Q(x, \beta)$$

with $\beta$ a vector of some real parameters. The quantile function $Q(x, \beta)$ is to be an increasing function of the risk level $\alpha$. The quantile estimators have to fulfil the monotonicity property with $\alpha$. Thus, a well specified quantile model is expected to provide estimators that behave like true quantiles and to increase for any values of parameters and conditioning variables according to the risk level $\alpha$. Quantile functions follow other properties as for instance the following examples.

1) If $f$ is a quantile function defined on $[0, 1]$ then:

$$Q(\alpha) = -f(1 - \alpha) \quad \text{and} \quad Q(\alpha) = f(\alpha^a)$$

with $a>0$ are also quantile functions, denoted $Q(\alpha)$.

2) If $f_{k=1,2,\ldots,n}$ are quantile functions with identical real domains, then:

$$Q(\alpha) = \sum_{k=1}^{K} a_k f_k(\alpha)$$

where $a_k$ are positive is also a quantile function.

These properties can be used for derive new quantile functions from an existing one.
We review below dynamic quantile models that have already been developed in the literature. Engle and Manganelli (2004) classify VaR calculation methods into three different categories: parametric, semi-parametric and non-parametric approaches.

For estimating a predetermined quantile, the most widely used non-parametric method is based on so-called historical simulations. These do not require any specific distributional assumptions and lead to estimate the VaR as the quantile of the empirical distribution of historical returns from a moving window of the most recent period. The essential problem is to determine the width of this window: including too few observations will lead to large sampling error, while using too many observations will result in estimates that are slow to react to changes in the true distribution of financial returns. Other methods involve allocating to the sample of returns exponentially decreasing weights (which sum to one). The returns are then ordered in ascending order and starting at the lowest return the weights are summed until the given confidence rate is reached; the conditional quantile estimate is set as the return that corresponds to the final weight used in the previous summation. The forecast is then constructed from an exponentially weighted average of past observations. If the distribution of returns is changing relatively quickly over time, a relatively fast exponential decay is needed to ensure coherent adapting. These exponential smoothing methods are simple and popular approaches.

Parametric approaches of quantile estimation involve a parameterization of the stock prices behavior. Conditional quantiles are estimated using a conditional volatility forecast with an assumption for the shape of the distribution, such as a Student-t. A GARCH model can be used for example to forecast the volatility (see Poon and Granger, 2003), though GARCH misspecification issues are well known.

Semi-parametric VaR approaches are based on Extreme Value Theory and Quantile Regression methods. Quantile regression estimation methods do not need any distributional assumptions. Conditional AutoRegressive VaR (CAViaR) introduced by Engle and Manganelli (2004) is one of them. They have defined directly the dynamics of risk by means of an auto
regression involving the lagged-Value at Risk (VaR) and the lagged value of endogenous variable called CAViaR. They present four CAViaR specifications: model with a symmetric absolute value, an asymmetric slope, indirect GARCH(1,1) and an adaptive form, denoted respectively: $C_{SVAR}(r_t;\beta)$, $C_{PA}(r_t;\beta)$, $C_{IG}(r_t;\beta)$, $C_{A}(r_t;\beta)$ where:

$$
\begin{align*}
C_{SVAR}(r_t;\beta) &= \beta_1 + \beta_2 \times C_{SVAR_{t-1}}(r_{t-1};\beta) + \beta_3 \times |r_{t-1}| \\
C_{PA}(r_t;\beta) &= \beta_1 + \beta_2 \times C_{PA_{t-1}}(r_{t-1};\beta) + \beta_3 \times \max(0,r_{t-1}) + \beta_4 \times \left[-\min(0,r_{t-1})\right] \\
C_{IG}(r_t;\beta) &= \left[\left[\beta_1 + \beta_2 \times C_{IG_{t-1}}(r_{t-1};\beta)\right]^2 + \beta_3 \times r_{t-1}^2\right] \\
C_{A}(r_t;\beta) &= C_{A_{t-1}}(r_{t-1};\beta) + \beta_1 \left[1 + \exp\left\{0.5 \times \left[r_{t-1} - C_{A_{t-1}}(r_{t-1};\beta)\right]\right\}\right]^{-0.01}
\end{align*}
$$

and where $\beta_i$ are parameters to estimate and $r_t$ is the risky asset return at time $t$.

Discussing the general CAViaR model without the autoregressive component, the conditional quantile function is well defined if parameters can be considered as quantile functions too. In fact, CAViaR models weight different baseline quantile functions at each date and can be therefore considered as quantile functions. Adding a non-negative autoregressive component of VaR, the CAViaR conditional quantile function becomes a linear combination of quantile functions weighted by non-negative coefficients. Thus CAViaR model satisfies the properties of a quantile function, even if the indirect GARCH(1,1) or adaptive specifications do not satisfy the monotonicity property. CAViaR model parameters are estimated using the quantile regression minimization (denotes QR Sum) presented by Koenker and Bassett (1978):

$$
\min_{\beta} \sum_i \left\{y_i - Q_i(\alpha)\left[\alpha - I_{y_i < Q_i(\alpha)}\right]\right\}
$$

with $Q_i(\alpha) = x_i^T \beta$ where $x_i$ is a vector of regressors, $\beta$ is a vector of parameters and $I_{\{\cdot\}}$ is an Heaviside function.

When the quantile model is linear, this minimization can be formulated as a linear program for which the dual problem is conveniently solved. Koenker and Bassett (1978) show that the resulting quantile estimator, $\hat{Q}_i(\alpha)$, essentially partitions the $y_i$ observations so that the proportion less than the corresponding quantile estimate is $\alpha$. 

The procedure proposed by Engle and Manganelli (2004) to estimate their CAViaR models is to generate vectors of parameters from a uniform random number generator between zero and one, or between minus one and zero (depending on the appropriate sign of the parameters. For each of the vectors then evaluated the QR Sum. The ten vectors that produced the lowest values for the function are then used as initial values in a quasi Newton algorithm. The QR sum is then calculated for each of the ten resulting vectors, and the vector producing the lowest value of the QR Sum is to be chosen as the final parameter vector.

Gouriéroux and Jasiak (2006) have introduced dynamic quantile models (DAQ). This class of dynamic quantile models is defined by:

\[ Q_t(\alpha, \beta) = \sum_{k=1}^{K} a_k(y_{t-1}, \beta_k) \times Q_k(\alpha, \beta_k) + a_0(y_{t-1}, \beta_0) \]  \( (10) \)

where \( Q_k(\alpha, \beta_k) \) are path-independent quantile functions and \( a_k(y_{t-1}, \beta_k) \) are non-negative functions of the past returns and other exogenous variables. A linear dynamic quantile model is linear in the parameters, then:

\[ Q_t(\alpha, \beta) = \sum_{k=1}^{p} \beta_k Q_{k,t}(\alpha) \]  \( (11) \)

Thus DAQ model can use different quantile functions to model a given quantile. To increase the accuracy of the model of the multiple, we can also combine different quantile functions, in a multi-quantile method. Actually, previous quantile functions can be extended to define a simple class of parametric dynamic quantile models.
4. The Time-varying Proportion Portfolio Insurance Approach: Some Empirical Evidences

In this section, we describe implementation methods for the conditional multiple model, presented in section 3, then we compare the performances of cushioned portfolio using several VaR estimation methods to compute the conditional multiple.

4.1 Conditional Multiples: Empirical Evidences

The sample period used in our study consisted of 29 years of daily data of the Dow Jones Index, from 2 January 1987 to 20 May 2005. This period delivered 4,641 returns. We have used a rolling window of 3,033 returns to estimate dynamically method parameters. The post sample period consisted of 1,608 returns. Figure 1 presents the data that are used for the empirical analysis.

Figure 1: Dow Jones Daily Returns

The first model of conditional multiple can be interpreted as:

\[ m_t = |VaR_{99\%} + d|^{-1} \]  

where \( VaR_{99\%} \) is the return first centile\(^4\), and \( d \) is a constant\(^5\).

\(^4\) The probability of 1% associated to the VaR was chosen in order for focusing on extremes and having at the same time enough points for backing-out good estimations – see below. It is also a standard in Risk Management for defining extreme loss. We tested several higher probability levels in the following; in these cases, the multiple is always flatter and its evolution is mostly explained by the variation of \( d \). In that sense, the introduction of \( d \) has the
The multiple is then given at each date by the inverse of the $VaR_{99\%}$ augmented by the exceeding maximum return during the estimation period. Thus the parameter $d$ allows taking into account the risky asset dispersion of return in the tail of the distribution, and especially for fat tailed distribution of the risky asset. Using this model the portfolio $VaR_{99\%}$ is controlled and extreme returns are taken into consideration.

To compute this conditional multiple, we use main methods of VaR estimation exposed in the literature or used by operational. Eight methods of VaR calculation are presented: one non-parametric method using the “naive” historical simulation approach, three methods based on distributional assumptions, and the four CAViaR specifications.

The $VaR_{99\%}$ based on historical (or “naive”) simulation is denoted $H_{VaR_{99\%}}$, it represents the empirical centile of in sample returns. $VaR_{99\%}$ based on distributional assumptions are the normal $VaR_{99\%}$ (denoted $Normal_{VaR_{99\%}}$), the Riskmetrics $VaR_{99\%}$ (denoted $RM_{VaR_{99\%}}$), and a GARCH(1,1) $VaR_{99\%}$ (denoted $GARCH_{VaR_{99\%}}$). The normal VaR is built under assumption of normality of returns. It’s computed with the empirical mean and standard deviation of the returns of the in sample period. The Riskmetrics VaR is a standard for practitioners: an exponential moving-average is used to forecast the volatility, and to compute the quantile assuming a Gaussian distribution. Another very popular method based on volatility forecasts is presented as the GARCH VaR. To compute it, we implement a GARCH(1,1) model. Our choice of the (1,1) specification was based on the analysis of the initial in-sample period of 3,033 returns and on the popularity of this order for GARCH models. We have derived the model parameters using maximum likelihood based on a Student-t distribution and the empirical distribution of effect of reducing the aggressivity of the strategy. Moreover, since the constant $d$ represents the highest failure of the model, corresponding to one of the highest negative return in the sample, the combination of the VaR and of $d$ is closely linked to a measure of the expected shortfall (see also conclusion).  

If we do not introduce the parameter $d$, the probability of violating the floor would have been equal to 1%. Working on a daily frequency, this probability would thus have been too high for describing a realistic investor’s demand (multiple often equal to 30 or so). However, tests with or without $d$ show no significant impact on the resulting global performance of the strategy on the sample.
standardized residuals. Finally, we present the multiple analysis, using the four CAViaR models (the Symmetric Absolute Value CAViaR, the Asymmetric Slope CAViaR, the IGARCH(1,1) CAViaR and the Adaptive CAViaR denoted respectively, SAV_CAViaR$_{99\%}$, AS_CAViaR$_{99\%}$, IGARCH_CAViaR$_{99\%}$, and Adaptive_CAViaR$_{99\%}$) using the procedure used by Engle and Manganelli (2004), the implementation method was already presented (see section 3).

Computing several VaR$_{99\%}$ methods (exposed above) for the 1,608 post sample period, we can see, on Table 3 and Figures 2, that except for the historical VaR$_{99\%}$ (using “naive” method and denoted H_VaR$_{99\%}$), and the VaR assuming a Gaussian distribution of the returns (denoted Normal_VaR$_{99\%}$) $^6$, the six other methods have a hit ratio not significantly different from 1%. According to this criterion, the conditional centile should be well modelled by these different methods. We can notice that the CAViaR asymmetric model has a hit ratio exactly equal to 1%, on the post sample period.

Figures 2: VaR$_{99\%}$ Estimates

**Fig.2.1**: Historical VaR$_{99\%}$

**Fig.2.2**: Risk metrics VaR$_{99}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, VaR are dynamically estimated for 1,608 post-sample periods; computations by the authors. Black lines represent the VaR evolution; the blue points represent the Dow Jones daily returns.

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$^6$ To evaluate the post sample conditional quantile estimates, we use the hit ratio. The hit ratio is the percentage of observations falling below the estimator. Ideally for a VaR$_{99\%}$, the percentage should be 1%. Here we have a sufficiently large sample, so the significance test can be performed on the percentage using a Gaussian distribution and the standard error formula for a proportion.
Fig.2.3: Normal VaR$_{99\%}$

Fig.2.4: GARCH(1,1) VaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, VaR are dynamically estimated for 1,608 post-sample periods; computations by the authors. Black lines represent the VaR evolution; the blue points represent the Dow Jones daily returns.

Fig.2.5: Symmetric Abs. Value CAViaR$_{99\%}$

Fig.2.6: Asymmetric Slope CAViaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, VaR are dynamically estimated for 1,608 post-sample periods; computations by the authors. Black lines represent the VaR evolution; the blue points represent the Dow Jones daily returns.

Fig.2.7: IGARCH CAViaR$_{99\%}$

Fig.2.8: Adaptive CAViaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, VaR are dynamically estimated for 1,608 post-sample periods; computations by the authors. Black lines represent the VaR evolution; the blue points represent the Dow Jones daily returns.
Table 1: Hit Percentage of 1% Conditional Quantile

<table>
<thead>
<tr>
<th>Hit ratio</th>
<th>H VaR</th>
<th>RM VaR</th>
<th>Normal VaR</th>
<th>GARCH VaR</th>
<th>SAV CAViaR</th>
<th>AS CAViaR</th>
<th>IGARCH CAViaR</th>
<th>Adaptive CAViaR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.49%*</td>
<td>1.24%</td>
<td>2.49%*</td>
<td>0.87%</td>
<td>0.87%</td>
<td>1.00%</td>
<td>1.12%</td>
<td>1.06%</td>
</tr>
</tbody>
</table>

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, VaR are dynamically estimated for 1,608 post-sample periods. The stars indicate hit ratios significantly different from 1%; computations by the authors. H VaR, RM VaR, Normal VaR, GARCH VaR, SAV VaR, AS CAViaR, IGARCH CAViaR, Adaptive CAViaR stand respectively for Historical VaR, Risk metrics VaR, the Gaussian VaR, the GARCH (1,1) VaR, the Symmetric Absolute Value CAViaR, the Asymmetric Slope CAViaR, the IGARCH CAViaR and the Adaptive CAViaR.

After having evaluated these $\text{VaR}_{99\%}$ models, we can use them for estimating conditional multiples according to the first model presented above:

$$m_i = \left| \text{VaR}_{i,99\%} + d \right|^{-1}$$

where $\text{VaR}_{i,99\%}$ is the return first centile, and $d$ is a constant.

We use the same notations, which were used for $\text{VaR}_{99\%}$ models. Thus for example, the conditional multiple estimated using the GARCH(1,1) $\text{VaR}_{99\%}$ (denoted GARCH_VaR$_{99\%}$) is denoted $m_{\text{GARCH}}$. Conditional multiple estimations over the 1,608 post sample periods are represented on figures 3. We note that, in October 1999, every conditional multiple flight from a level around 1.5, to 4. It is due to the exit of October 1987 crisis from the estimation period. The estimations of conditional multiples spread between 1.5, and 6, which are compatible with multiple values used by practitioners on this market (between 3 and 8). The multiple is the parameter which determines the exposition of the cushioned portfolio. To guarantee a predetermined floor, the multiple has to be inferior to the inverse of the potential loss that the risky asset could reach before the portfolio manager rebalances its position. If we assume that the portfolio manager can rebalance totally his position during one day, all estimations of conditional multiple with this first model allow guaranteeing the predetermined floor defined by the investor.
**Figures 3:** Conditional Multiple based on VaR$_{99\%}$ Estimates

**Fig. 3.1:** Multiple based on the Historical VaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.

**Fig. 3.2:** Multiple based on the Risk metrics VaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.

**Fig. 3.3:** Multiple based on the Normal VaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.

**Fig. 3.4:** Multiple based on the GARCH(1,1)VaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.

**Fig. 3.5:** Multiple based on the Symmetric Absolute Value CAViaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.

**Fig. 3.6:** Multiple based on the Asymmetric Slope CAViaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.
Fig. 3.7: Multiple based on the IGARCH CAViaR$_{99\%}$

Fig. 3.8: Multiple based on the Adaptive CAViaR$_{99\%}$

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors.

| Table 2: Under-estimation of the Max Drawdown by Conditional Multiples |
|--------------------------|----------------|----------------|----------------|----------------|-------------|----------------|----------------|
| Under estimations       | M_H          | m_RM          | m_Normal       | m_GARCH        | m_SAV        | m_AS         | m_IGARCH       | m_Adaptive     |
|                         | never        | never         | never          | never          | never        | never        | never          | never          |

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors. The notations m_H, m_Risk_metrics, m_Normal, m_GARCH, m_SAV, m_AS, m_IGARCH, m_Adaptive stand respectively for multiple based on Historical VaR, Risk metrics VaR, the Gaussian VaR, the GARCH (1,1) VaR, the Symmetric Absolute Value CAViaR, the Asymmetric Slope CAViaR, the IGARCH CAViaR, and the Adaptive CAViaR.

If we analyze the distributions of conditional multiple (see figure 4), we observe that for every conditional multiple estimation methods we obtain two modes. One mode is around a low value of the multiple (between 1 and 2 depending of estimation methods) associated with a more defensive behavior of the cushioned portfolio during turbulent period, and another mode around 4.5 associated to “classical” and realistic value of the multiple. On one hand, densities of conditional multiple using Asymmetric Slope, IGARCH(1,1) and Symmetric Value of the multiple are very similar, and on the other hand “naive” historical, adaptive and Gaussian conditional multiples densities have comparable characteristics. For the sake of clarity, we have thus only represented on figure 4 the densities of the asymmetric slope model and for the “naive” historical estimation of the multiple.
4.2. Conditional versus Unconditional Strategy Comparison Evidences

Values of cushioned portfolio are path dependent. The performances of these guaranteed portfolios, determined by using different estimation methods of the conditional multiple, are not easy to compare. Actually, the performance of the insured portfolio depends more on its start date, and on the investment horizon than of the estimation of the conditional multiple. To investigate whether these different estimations methods lead to significantly different performances (for a long term analysis), we propose to introduce an original “multi-start” analysis. The “multi-start analysis” consists for a fixed investment horizon (here one year) in computing every value of insured portfolios beginning at every moment of the post sample period, according to its conditional multiple estimations. An illustration of the method is presented on figure 5. Table 3 reports the main characteristics of the returns of cushioned portfolio using the “multi-start” analysis.
On the contrary of what we observe in a classical analysis with a predefine start and end of analysis, the characteristics of covered portfolios returns are not significantly different. But if we choose a specific start and end date, then covered portfolio returns using this conditional multiple can be more different.

In following sections, the analysis is limited to the Asymmetric Slope multiple model, for the sake of simplicity and because of the similarity of the behavior of this multiple with the one of the best multiple estimation methods. The Asymmetric Slope multiple based Time-varying
Proportion Portfolio Insurance was dynamically back-tested from 1937: the floor was violated only one time (during the 1987 crash), loosing 0.61% of its guaranteed value.

**Table 4:** Extended Backtest of the Asymmetric Slope TVPPI Multiple from 1937 on the US Market

<table>
<thead>
<tr>
<th>Conditional Asymmetric Slope Multiple TVPPI</th>
<th>Violating Floor (date)</th>
<th>Amplitude of Floor Violation</th>
<th>Annualized Excess Return</th>
<th>Volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPPI Multiple 1</td>
<td>Never</td>
<td>-</td>
<td>0.09%</td>
<td>0.34%</td>
</tr>
<tr>
<td>CPPI Multiple 2</td>
<td>Never</td>
<td>-</td>
<td>0.19%</td>
<td>0.68%</td>
</tr>
<tr>
<td>CPPI Multiple 3</td>
<td>Never</td>
<td>-</td>
<td>0.31%</td>
<td>1.05%</td>
</tr>
<tr>
<td>CPPI Multiple 4</td>
<td>One time (20/10/1987)</td>
<td>0.05%</td>
<td>0.45%</td>
<td>1.48%</td>
</tr>
<tr>
<td>CPPI Multiple 5</td>
<td>One time (20/10/1987)</td>
<td>0.47%</td>
<td>0.60%</td>
<td>1.97%</td>
</tr>
<tr>
<td>CPPI Multiple 6</td>
<td>One time (20/10/1987)</td>
<td>0.78%</td>
<td>0.77%</td>
<td>2.53%</td>
</tr>
<tr>
<td>CPPI Multiple 7</td>
<td>One time (20/10/1987)</td>
<td>0.97%</td>
<td>0.96%</td>
<td>3.18%</td>
</tr>
<tr>
<td>CPPI Multiple 8</td>
<td>One time (20/10/1987)</td>
<td>1.05%</td>
<td>1.16%</td>
<td>3.90%</td>
</tr>
</tbody>
</table>

Source: Bloomberg, Daily Returns of the Dow Jones Index from 10/02/1928 to 12/21/2005, conditional multiple is every week dynamically estimated; computations by the authors. Excess returns are calculated against the risk free rate. A successive one year investment capital guarantee horizon was used.

5. About the Gap Risk Estimation

We have used a probabilistic approach to model the conditional multiple. By construction, the guarantee is associated to a tiny level of probability but it is not an absolute guarantee. The risk of violating the floor protection is called gap risk.

At any rate, even in an unconditional multiple CPPI framework, the guarantee depends on the estimation of the maximum potential loss that the risky asset can reach before the portfolio manager is able to rebalance his position. Actually, in continuous time, the CPPI strategies provide a value above a floor level unless the price dynamic of the risky asset has jumps. In practice, it is caused by liquidity constraints and price jumps. Both can be modelled in a setup where the price dynamic of the risky asset is described by a continuous–time stochastic process but trading is restricted to discrete time. If the potential loss is underestimated, the predefined guarantee of the portfolio is not anymore insured. The only way to be sure to reach a perfect guarantee is to choice an unconditional multiple equal to one (the potential loss is then
supposed to be 100%) For all other cases, we should estimate the gap risk between a perfect insurance and the insurance proposed assuming an estimation of the potential loss. We propose here a way to estimate this gap risk, and we recommend to add it as an additional performance criterion of insured portfolios.

To estimate the gap risk in a CPPI framework, we use the multiple at any time to get the assumed estimation of the maximum potential loss. To reach a perfect guarantee assuming no maximum potential loss scenario, we will hedge the gap risk buying a put whose maturity will be the rebalance frequency. The strike will be defined each day thanks to the assumed (or modelled) maximal potential loss at $T$. The put price is computed using several pricing methods for European options and realistic transaction costs with the data used in previous section (for the Asymmetric Slope model).

<table>
<thead>
<tr>
<th>Options Model used to estimate the Gap Risk:</th>
<th>Black and Scholes' Model</th>
<th>Cox, Ross, Rubinstein's Binomial Tree</th>
<th>Boyle's Trinomial Tree</th>
<th>Merton's Jump Diffusion Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fees (in basis points)</td>
<td>Annual Gap Risk Costs (in basis points)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>70.00</td>
<td>46.18</td>
<td>46.18</td>
<td>46.18</td>
<td>46.21</td>
</tr>
<tr>
<td>80.00</td>
<td>52.69</td>
<td>52.69</td>
<td>52.69</td>
<td>52.71</td>
</tr>
<tr>
<td>90.00</td>
<td>59.17</td>
<td>59.17</td>
<td>59.17</td>
<td>59.19</td>
</tr>
<tr>
<td>100.00</td>
<td>65.63</td>
<td>65.63</td>
<td>65.63</td>
<td>65.65</td>
</tr>
<tr>
<td>120.00</td>
<td>78.48</td>
<td>78.48</td>
<td>78.48</td>
<td>78.51</td>
</tr>
<tr>
<td>140.00</td>
<td>91.25</td>
<td>91.25</td>
<td>91.25</td>
<td>91.27</td>
</tr>
<tr>
<td>200.00</td>
<td>129.04</td>
<td>129.04</td>
<td>129.04</td>
<td>129.06</td>
</tr>
</tbody>
</table>

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computations by the authors. No difference can be made between the three first option models when pricing the insured portfolios gap risk from 01/02/1987 to 05/25/2005.
Figure 7: Estimation of the Gap Risk using a Jump Diffusion Model for the Asymmetric Slope Model of the Multiple

Source: Datastream, Daily Returns of the Dow Jones Index from 01/02/1987 to 05/25/2005, conditional multiples are dynamically estimated for 1,608 post-sample periods; computation by the authors. The insured portfolios gap risk is estimated from 01/02/1987 to 05/25/2005 using the Asymmetric Slope Model of the Multiple.

6. Preliminary Conclusions

The model proposed in this paper for the conditional multiple, allows guaranteeing the insured floor according to market evolutions, on a large post-sample period. This method provides a rigorous framework to determine the multiple, which is the main parameter of cushioned portfolio preserving a constant exposition to risk. To compute the conditional multiple, according to this model, an appropriate method to estimate the centile of the risky asset return has to be chosen. If the centile is well modelled (hit ratio not significantly different from 1% and no cluster of exceedances), the guarantee is insured. Moreover, even if huge differences of characteristics exist between the returns of covered portfolio using this model and different
centile specifications for specified period, no significant differences on long-term performance can be underlined over the large post sample period, by the “multi-start” analysis.

We model in the paper the multiple as a function of the conditional centile. A first natural extension of our work will consist in replacing this reference by the Expected Shortfall expressed in a quantile regression conditional setting. This would have the advantages of dealing with a more robust and flexible multiple centile estimations at the same time, in a coherent risk measure framework. Moreover, this paper has introduced the use of an original “multi-start/horizon” performance comparison framework, which seems more adapted to path and start dependent financial product such as the constant proportion portfolio insurance portfolio. This comparison framework will be thus second further extended with the introduction of the Multi-start Time-varying Proportion Portfolio Insurance Approach (MTPPI) based on aggregation of various multi-start TPPI. Thirdly, we have also proposed an empirical way for estimating the gap risk between theoretical perfect portfolio insurance and the insurance proposed with transaction costs. Additional performance criteria will also be introduced to examine the adequacy of such insured portfolios together with the multi-quantile multi-start approach.
References


Appendix 1: CPPI based on Quantile Criterion (in a Marked Point Process Framework)

A “guaranteed” portfolio is defined so that the portfolio value will be always above a predefined floor at a given high probability level. Assume that the risky price follows a marked point process, which is characterized by the sequence of marks \((X_t)\) and the increasing sequence of times \((T_t)\) at which the risky asset varies.

In the CPPI framework, the first following “global” quantile hedging condition can be considered (see Bertrand and Prigent, 2002):

\[
P\left[ \forall t \leq T, C_t \geq 0 \right] \geq 1 - \delta
\]

which is equivalent to:

\[
m \leq \frac{1}{L_t^\top (1 - \delta)}
\]

where:

\[
L_t(x) = \sum_j P\left[ M_j \leq x \mid T_j \leq T < T_{t+1} \right] \times P\left[ T_j \leq T < T_{t+1} \right]
\]

and:

\[
M_t = \text{Max}\left\{ \frac{S_j - S_0}{S_0}, \ldots, \frac{S_j - S_{t-1}}{S_{t-1}} \right\}.
\]

Another “local” quantile condition can also be introduced, based on conditional quantile:

\[
\forall t \leq T, P_t\left[ C_t > 0 \mid C_{t-1} > 0 \right] \geq 1 - \alpha
\]

where \(P_t\) denotes the conditional probability given the observation of the marked point process until time \(t\). More precisely, for \(t \in [T_i, T_{i+1}]\), we have:

\[
P_t\left[ C_t > 0 \mid C_{t-1} > 0 \right] = P\left[ C_{T_i} > 0 \mid G_{T_{i+1}} \right]
\]

where \(G_{T_{i+1}}\) is the \(\sigma\)-algebra generated by the set of all intersections of \(\{C_{T_{i+1}} > 0\}\) with any subset \(A_{T_{i+1}}\) of the \(\sigma\)-algebra generated by the observation of the marked point process until time \(T_{i+1}\).
From previous condition, an upper bound on the multiple can be deduced, according to specific assumptions (see Ben Hameur and Prigent, 2007) for the special case of GARCH-type models with deterministic transaction times.

Appendix 2: A Continuous Time Version of the TPPI Strategy

We consider a filtered probabilistic space \([\Omega, \mathcal{F}, (\mathcal{F}_t), P]\). We assume that the risky asset price follows a diffusion process:

\[
dS_t = S_t [\mu(t, S_t) dt + \sigma(t, S_t) dW_t],
\]

where \(W\) is a standard Brownian motion and the functions \(\mu(\cdot, \cdot)\) and \(\sigma(\cdot, \cdot)\) satisfy usual Lipschitz and Growth conditions to ensure the existence and unicity of the solution of the previous stochastic differential equation.

The CPPI portfolio value satisfies:

\[
dV_{t,CPP} = (V_{t,CPP} - e_t) \frac{dB_t}{B_t} + e_t \frac{dS_t}{S_t} + dF_t,
\]

with:

\[
V_{t,CPP} = C_t + F_t,
\]

\[e_t = m_t C_t,\]

\[
\frac{dF_t}{F_t} = \frac{dB_t}{B_t} = rd_t.
\]

Case 1: The multiple \(m_t\) evolves in continuous-time.

It is assumed to follow a predictable stochastic process such that next Condition (E) is satisfied:

\[
\int_0^T m_s ds < +\infty \text{ and } \int_0^T m_s^2 ds < +\infty, \text{ almost surely.}
\]

Then, the cushion value \(C\) satisfies:

\[
dC_t = d(V_{t,CPP} - F_t),
\]

\[
= (V_{t,CPP} - e_t) \frac{dB_t}{B_t} + e_t \frac{dS_t}{S_t} - dF_t,
\]

\[
= (C_t - m_t C_t) \frac{dB_t}{B_t} + (m_t C_t) \frac{dS_t}{S_t},
\]

\[
= C_t \{r + m_t [\mu(t, S_t) - r] dt + m_t \sigma(t, S_t) dW_t \}.\]
Under the previous Condition (E), we deduce:

\[ V_t^{CPPI} = C_t + F_0 e^{rt}, \]

with

\[ C_t = C_0 \exp \left\{ \int_0^t \left\{ r + m_s \left[ \mu (s, S_s) - r \right] - \frac{1}{2} m_s^2 \sigma^2 (s, S_s) \right\} ds + \int_0^t m_s \sigma (s, S_s) dW_s \right\}. \]

Case 2: The multiple and the exposition can only be changed on a discrete-time basis (more realistic case).

Consider the times \( t_0 < t_1 < \ldots < t_n = T \). Both the cushion and the exposition are fixed during each time period \( [t_i, t_{i+1}] \) (since they are simple processes). Then, the multiple, which is equal to the ratio \( e_t / C_t \), is also fixed on each \( [t_i, t_{i+1}] \).

Therefore, we get, for any time \( t \in [t_i, t_{i+1}] \):

\[ V_t = (V_{t_i} - e_{t_i}) \exp \left[ r(t - t_i) \right] + e_{t_i} \exp \left\{ \int_{t_i}^t \left\{ \mu (s, S_s) - \frac{1}{2} \sigma^2 (s, S_s) \right\} ds + \int_0^t \sigma (s, S_s) dW_s \right\}, \]

with:

\[ e_{t_i} = m_{t_i} \left\{ V_{t_i} - F_0 \exp \left[ r(t_i - t_0) \right] \right\}. \]

Thus, we can deduce the portfolio value \( V_t \) by induction.

When the risky asset price follows a geometric Brownian motion (i.e. \( \mu (\cdot, \cdot) \) and \( \sigma (\cdot, \cdot) \) are constant), we get:

\[ P \left[ C_{t_{i+1}} > 0 \bigg| C_{t_i} > 0 \right] \geq (1 - \alpha) \]

\[ \iff m \leq \frac{1}{1 - \exp\left[ N^{-1}(\alpha) \sigma (t_{i+1} - t_i)^{-1/2} + \left( \mu - r - 1/2 \sigma^2 \right) (t_{i+1} - t_i) \right]} \]

where \( N^{-1}(\alpha) \) denotes the \( \alpha \)-quantile of the standard Gaussian distribution.