Reduced form modelling for credit risk

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Abstract

The purpose of this paper is to present in a unified context the reduced form modelling approach, in which a credit event is modelled as a totally inaccessible random time. Once the general framework is introduced (frequently referred to as the "pure intensity" set-up), we focus on the special case where the full information at the disposal of the traders may be split in two sub-filtrations, one of them carrying the full information of the occurrence of the credit event (framework often referred to as "hazard process" approach). When only one filtration is considered, the general pricing rule reveals to be non tractable in most cases, whereas the second construction leads to much simpler formulas. Examples are given and evidence brought that this set-up is more tractable.

Introduction

Given the flow of information of a financial market, containing both defaultable and default free assets, the methodology for modelling a credit event can be split into two main approaches: The structural approach (chronologically the first one) and the reduced form approach.

- In the structural framework, the credit event is modelled as the hitting time of a barrier by a process adapted to the information flow (typically the value of the firm crossing down a debt ratio). This approach is intuitive (directly linked to economic fundamentals, such as for example the structure of the balance sheet of the company), and the valuation and hedging theory relies on tools very close to the techniques involved in the classical Black and Scholes default-free set up.
  Nonetheless it presents important drawbacks: the value process can not be easily observed, it is not a tradeable security, and, most of all, a relevant trigger is very complex to identify. A simple continuous firm's value process implies a predictable credit event, leading to unnatural features such as null spreads for short maturities.

  The interested reader may refer to the ground articles of [4], [32] and [13], to [30] for the introduction of constant barrier and random interest rates, to [25] for the introduction of random barriers or random interest rates, to [29], [28], [17] and [8] for a study of optimal capital structure, bankruptcy costs and tax benefits, and to [35], [17] and [38] for discontinuous firm's value process examples (framework that does not imply null spreads at short maturities).

- The reduced form approach lies on the assumption that the credit event occurs by "surprise", i.e., at a totally inaccessible time and consists in the modelling of the conditional law of this random time (see later). Of course, the intensity of the credit event occurence should be dependent of structured factors (as stock levels or interest rates).

  The present paper will mainly focus on this framework. An example of transformation of a structural model into a reduced form model will be studied in the third section in connection with the paper of Guo et al. (see [16]).

In the literature, reduced form framework has been split so far into two different approaches, depending on wether the information of the default free assets - a sub-filtration of the filtration containing the whole information of the financial market - was introduced or not (the former being referred to as "hazard process approach" and the later as "intensity approach").

- In the "Intensity based approach", a unique flow of information is considered, and the credit event is a stopping time of this filtration. The modelling is based on the existence of "an intensity rate process": a non-negative process satisfying a compensation property (cf. first section below). Classical methods allow to compute this process, and to derive a pricing rule for conditional claims (see [10]).
  The main problem in this methodology is that the pricing rule (referred to in the sequel as "intensity based pricing rule", $IBPR$) leads to a non tractable formula, involving computations complex to handle[1].

- The second approach is based on the computation of the "hazard process" and lies on the introduction of two filtrations: a reference filtration, enlarged by the progressive knowledge of the credit event occurence. This framework allows to derive a pricing rule much more convenient to use (referred to in the sequel as "hazard based pricing rule", $HBPR$). However, it depends on the assumption of the existence of a decomposition between the credit event and another filtration (a "default-free market" information is often mentioned). This modelling assumption is particularly meaningful when the default free market contains information not depending on the credit event, such as the stochastic interest rates for instance (see [2]). The filtration enlargement method was first used by Lando (see [27]) in its construction of Cox processes in a pure intensity approach, before being reintroduced for the definition of the hazard process modelling.

The main goal of this paper will be the presentation of the reduced form framework, in theory and practice. It will focus mainly on the filtration enlargement approach, presented as a particular case of the more general "intensity based approach", and leading to more efficient pricing tools.

The paper is organized as follows: in a *first section*, we present the two approaches, and the techniques and results relative to each one. Our point is to emphasize that the hazard process framework is the more tractable in many features, notably for pricing. This simplicity has nonetheless a cost: this setting, based on filtration enlargements, presents some technical constraints inherent in this mathematical theory. Indeed, once specified the dynamics of the default-free assets, a special care must be brought to the changes due to the new information: hypotheses has to be made on the nature of the random time modelling the credit event so that to preserve the semi-martingale properties (invariance feature called $(\mathcal{H}')$ hypothesis). The *second section* deals with these aspects (and presents applications of the initial times[2] in this

[1] The formula involves quantities such as the jump at the credit event of some stochastic process
[2] see Section 2.1, [37] and [19] for definition and properties of these random times

context). We also focus within this part on the question of $(\mathcal{H})$ hypothesis, under which the martingales of the small filtration remain martingales in the full filtration. This property of the progressive enlargement of the reference filtration by the random time - also called immersion - is often a central feature asked to the model[3]. Under this hypothesis the stopped hazard process is the compensator of the credit event (the intensity process), and the two pricing rule are very close in interpretation. The *third section* is dedicated to examples of meaningful models in which immersion does not hold. In such cases $IBPR$ may involve a non null jump and reveal to be non tractable, and the pricing should be based on $HBPR$. The *last section* presents pricing examples, based on defaultable zero coupons and credit default swaps.

The following notation will be used in the sequel: for a given filtration $\mathbb{F}$ and probability $\mathbb{P}$, the set $\mathcal{M}(\mathbb{F},\mathbb{P})$ (resp. $\mathcal{S}(\mathbb{F},\mathbb{P})$) denotes the set of $(\mathbb{F},\mathbb{P})$-martingales (resp. $(\mathbb{F},\mathbb{P})$-semi-martingales). When there is no confusion with the choice of probability $\mathbb{P}$, we write $\mathcal{M}(\mathbb{F})$ for $\mathcal{M}(\mathbb{F},\mathbb{P})$. We denote by $\mathcal{P}(\mathbb{F})$ the set of $\mathbb{F}$-predictable processes. For a filtration enlargement $\mathbb{F} \subset \mathbb{G}$, we say that $(\mathcal{H})$ hypothesis holds if $\mathcal{M}(\mathbb{F}) \subset \mathcal{M}(\mathbb{G})$ (and write $\mathbb{F} \hookrightarrow_{(\mathcal{H})} \mathbb{G}$), and that $(\mathcal{H}')$ hypothesis holds if $\mathcal{S}(\mathbb{F}) \subset \mathcal{S}(\mathbb{G})$ (and write $\mathbb{F} \hookrightarrow_{(\mathcal{H}')} \mathbb{G}$).

1 The two approaches of reduced form modelling

We present here the "intensity framework", and the "hazard process framework", the two main approaches in reduced form modelling, and emphasize that the second, based on an enlargement of filtration, is a particular case of the first one, and offers easier formulas for pricing (see the second section for the links between $IBPR$ and $HBPR$). Cox process construction, the classical method for the construction of the credit event and the intensity process, is the simplest example of filtration enlargement construction (see the following section for examples). For a detailed presentation of these approaches, see [10], [12], [21], or [3].

1.1 Intensity based models

In intensity based models, the default time τ is a stopping time in a given filtration $\mathbb{G}$, representing the full information of the market.

The process $(H_t = \mathbb{1}_{\tau \leq t},\, t \geq 0)$ is a $\mathbb{G}$-adapted increasing càdlàg process, hence a $\mathbb{G}$-submartingale, and there exists a unique $\mathbb{G}$-predictable increasing process Λ, called the compensator, such that the process

$$M_t = H_t - \Lambda_t$$

is a $\mathbb{G}$-martingale. As $H = 0$ after default, its compensator has to be constant since the $\mathbb{G}$ -martingale M can not be decreasing after the $\mathbb{G}$-stopping time τ. It follows that the compensator satisfies $\Lambda_t = \Lambda_{t\wedge\tau}$. The process Λ is continuous if and only if τ is a $\mathbb{G}$-totally inaccessible stopping time. In intensity based models, it is generally assumed that Λ is absolutely continuous with respect to Lebesgue measure, i.e., that there exists a non-negative $\mathbb{G}$-adapted process $(\lambda^{\mathbb{G}}_t, t \geq 0)$ such that

$$M_t = H_t - \int_0^t \lambda^{\mathbb{G}}_s ds$$

is a martingale[4]. This process $\lambda^{\mathbb{G}}$ is called the intensity rate and vanishes after time τ.

We refer to [14] for cases where the absolute continuity assumption does not hold.

[3] for example when the reference market is complete without arbitrage opportunities

[4] Remark that the submartingale H is of class (D) - the familly $\{H_T,\, T$ any finite valued stopping time$\}$ is uniformly integrable - since it is bounded by one.

The classical way to compute the intensity rate is Aven's lemma [1], or the Laplacian approximation method, which gives (see for example Meyer [33]) an efficient tool to obtain the predictable bounded variation part A of a $\mathbb{G}$-semimartingale X (under technical conditions, for a counter example see [9]) as

$$A_t = \lim_{h\to 0} \int_0^t \frac{1}{h}\mathbb{E}(X_{s+h} - X_s|\mathcal{G}_s)ds$$

One gets, in a credit default setting, for $X_t = H_t$, under some regularity assumption,

$$\lambda_t^{\mathbb{G}} = \lim_{h\to 0} \frac{1}{h}\mathbb{P}(t < \tau \le t + h|\mathcal{G}_t),$$

when the limit exists.

For the pricing matter, we have for $X \in \mathcal{G}_T$, integrable,

$$\mathbb{E}(X1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\left(V_t - \mathbb{E}(\Delta V_\tau 1\!\!1_{\{\tau\le T\}}|\mathcal{G}_t)\right) \tag{1}$$

with

$$V_t = e^{\Lambda_t}\mathbb{E}(Xe^{-\Lambda_T}|\mathcal{G}_t) = e^{\Lambda_{t\wedge\tau}}\mathbb{E}(Xe^{-\Lambda_{T\wedge\tau}}|\mathcal{G}_t)$$

We shall refer to this formula in the sequel as the "intensity based rule", or $IBPR$. The detailed proof of this result can be found in (see [10]); the main idea is to apply the integration by part formula to the product $U = VL$ (remark $U_T = 1\!\!1_{\{T<\tau\}}X$), with $L_t = 1 - H_t$: $dU_t = \Delta V_\tau dL_t + (L_{t-}dm_t - V_{t-}dM_t)$, (where $dm_t = e^{\Lambda_t}dY_t$, for $Y_t = e^{-\Lambda_t}V_t$), which yields to $U_t = \mathbb{E}(\Delta V_\tau 1\!\!1_{t<\tau\le T} + U_T|\mathcal{G}_t)$. Using the intensity rate, the pricing rule becomes:

$$\mathbb{E}(X1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\mathbb{E}\left(Xe^{-\int_t^T\lambda_s^{\mathbb{G}}ds}\middle|\mathcal{G}_t\right) - \mathbb{E}(\Delta V_\tau 1\!\!1_{\{t<\tau\le T\}}|\mathcal{G}_t).$$

For example, whereas the price of a zero-coupon bond writes (if $\beta_t = \exp - \int_0^t r_s ds$ denotes the savings account):

$$B(t,T) = \beta_t\mathbb{E}\left(\frac{1}{\beta_T}\middle|\mathcal{G}_t\right) = \mathbb{E}\left(e^{-\int_t^T r_s ds}\middle|\mathcal{G}_t\right),$$

the price of a defaultable zero-coupon bond is:

$$D(t,T) = \beta_t\mathbb{E}\left(\frac{1\!\!1_{T<\tau}}{\beta_T}\middle|\mathcal{G}_t\right) = 1\!\!1_{\{t<\tau\}}\mathbb{E}\left(e^{-\int_t^T(r_s+\lambda_s^{\mathbb{G}})ds}\middle|\mathcal{G}_t\right) - \mathbb{E}(\Delta V_\tau 1\!\!1_{\{t<\tau\le T\}}|\mathcal{G}_t).$$

The main difficulty in that framework is the computation of the jump of the process V. As an illustration, let us present as a simple example the computation of the price of a defaultable zero-coupon bond when the process $(H_t - \lambda(t\wedge\tau), t\ge 0)$ is a martingale, where λ is a constant (i.e., τ is an exponential random variable with parameter λ) and where $r = 0$. The filtration $\mathbb{G}$ is here the filtration generated by the process H, hence $\mathcal{G}_t = \sigma(t\wedge\tau)$. A direct computation, based on the computation of conditional probability, shows that

$$\mathbb{E}(1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}e^{-\lambda(T-t)}.$$

The $\mathbb{G}$-adapted intensity rate is $\lambda_t = 1\!\!1_{t<\tau}\lambda$ and $e^{-\Lambda_{t\wedge\tau}} = e^{-\lambda(t\wedge\tau)}$. In order to compute $\mathbb{E}(1\!\!1_{T<\tau}|\mathcal{G}_t)$ using (1), we introduce $V_t = e^{\lambda(t\wedge\tau)}\mathbb{E}\left(e^{-\lambda(T\wedge\tau)}|\mathcal{G}_t\right)$. Then,

$$\begin{aligned} V_t &= \mathbb{1}_{t<\tau} e^{\lambda t} \frac{\int_t^\infty e^{-\lambda(u\wedge T)} f(u)du}{\int_t^\infty f(u)du} + \mathbb{1}_{\tau \le t} e^{\lambda\tau} e^{-\lambda\tau} \\ &= \mathbb{1}_{t<\tau} \left(\frac{1-e^{-2\lambda(T-t)}}{2} + e^{-\lambda(T-t)} \right) + \mathbb{1}_{\tau \le t} \end{aligned}$$

It follows the jump of V at time τ is non null and is equal to: $\Delta V_\tau = 1 - \left(1 - e^{-2\lambda(T-\tau)}\right)/2 - e^{-\lambda(T-t)}$. Then, one find, after some computations, that $\mathbb{E}\left(\mathbb{1}_{T<\tau}\,|\mathcal{G}_t\right) = \mathbb{1}_{\{t<\tau\}} e^{-\lambda(T-t)}$.

The next section presents the particular case where the full filtration can be split into two sub-filtrations. This framework allows to derive a second pricing formula (the $HBPR$), much more efficient than $IBPR$ (no jump part in the formula, no need to compute the intensity). For instance, its application to the previous example would lead immediately to the conclusion.

1.2 Hazard process models

The hazard process approach is based on the assumption that some reference filtration $\mathbb{F}$ is given (see for example Kusuoka [26] and Elliott et al. [12]). The default time is a random time, which is not an $\mathbb{F}$-stopping time. The filtration $\mathbb{G}$ is defined as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathbb{H}$ is the filtration generated by the process ($H_t = \mathbb{1}_{\tau \le t}, t \ge 0$), in particular τ is a $\mathbb{G}$-stopping time. This "separation" into two filtrations is often quite natural (see for example [20]). The simplest case is the Cox process model class we shall present in section 2.3. Other examples will follow below.

Let $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. We make the technical assumption that this process does not vanish (the case without this assumption has been treated in [2], see also the third section in the sequel). We have, for $X \in \mathcal{F}_T$, the very simple pricing rule (see [12]) - which does not involve the jump of any auxiliary process, nor the knowledge of the intensity:

$$\mathbb{E}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t} \mathbb{E}(G_T X|\mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\left(X e^{-\Gamma_T}\,|\mathcal{F}_t\right), \tag{2}$$

setting $\Gamma_t = -\ln G_t$. This process Γ is called the hazard process. We shall refer to this formula in the sequel as "Hazard based pricing rule" or $HBPR$.

The process $F = 1 - G$ is an $\mathbb{F}$-submartingale (first studied by Azéma) and admits a Doob-Meyer decomposition as $F_t = Z_t + A_t$ where Z is an $\mathbb{F}$-martingale[5] and A a predictable increasing process. In what follows, we write $\mathbb{F}$-Doob-Meyer decomposition in order to make precise the choice of the reference filtration. We introduce the $\mathbb{F}$-adapted increasing process $\Lambda^{\mathbb{F}}$ defined by

$$\Lambda_t^{\mathbb{F}} = \int_0^t \frac{dA_s}{G_{s-}}$$

It is easy to prove (integration by part formula) that the process ($M_t = H_t - \Lambda^{\mathbb{F}}_{t\wedge\tau}, t \ge 0$) is a $\mathbb{G}$-martingale, hence the uniqueness of the compensator implies that $\Lambda_t = \Lambda^{\mathbb{F}}_{t\wedge\tau}$.

Assume that the process A is absolutely continuous with respect to the Lebesgue measure ($dA_s = a_s ds$). Then,

$$\lambda_t^{\mathbb{G}} = \mathbb{1}_{t<\tau} \lambda_t^{\mathbb{F}} \quad \text{where} \quad \lambda_t^{\mathbb{F}} = \frac{a_t}{G_{t-}}.$$

[5] Remark that the submartingale F is of class (D), since bounded by one.

We shall say, with an abuse of language that $\lambda^{\mathbb{F}}$ is the $\mathbb{F}$-intensity rate[6].

It is important for the intuition of the meaning of the intensity and the "$\mathbb{F}$-intensity" to remark that:

$$\left\{\begin{array}{lcl}\lambda_t^{\mathbb{G}} & = & \lim_{h\to 0}\frac{1}{h}\mathbb{P}(t<\tau\le t+h|\mathcal{G}_t)\\ \lambda_t^{\mathbb{F}} & = & \lim_{h\to 0}\frac{1}{h}\frac{\mathbb{P}(t<\tau<t+h|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}\end{array}\right. \tag{3}$$

a formula which can be obtained via the Laplacian approximation, the increasing process associated with $(H_t, t\ge 0)$ being obtained as the limit when h goes to 0 of

$$\frac{1}{h}\int_0^t \mathbb{E}(H_{s+h}-H_s|\mathcal{G}_s)ds = \frac{1}{h}\int_0^t 1\!\!1_{s<\tau}\frac{\mathbb{E}(H_{s+h}-H_s|\mathcal{F}_s)}{\mathbb{P}(\tau>s|\mathcal{F}_s)}ds.$$

Comparing the two pricing rules (1) and (2), we have for every $X\in\mathcal{F}_T$ the pricing equality:

$$1\!\!1_{\{t<\tau\}}\left(V_t-\mathbb{E}(\Delta V_\tau 1\!\!1_{\{\tau\le T\}}|\mathcal{G}_t)\right) = 1\!\!1_{\{t<\tau\}}e^{\Gamma_t}\mathbb{E}(Xe^{-\Gamma_T}|\mathcal{F}_t)$$

with $V_t = e^{\Lambda_{t\wedge\tau}}\mathbb{E}(Xe^{-\Lambda_{T\wedge\tau}}|\mathcal{G}_t)$. Note that the computation of the left-hand side requires the Doob-Meyer decomposition of G and the computation of the jump of V, whereas the right-hand side requires only the knowledge of G.

Next section presents the $(\mathcal{H})$ hypothesis, where both framework are very close (the hazard process and the intensity process are the same until default, under the assumption that the default time avoids the $\mathbb{F}$-stopping times), and which can be a quite natural hypothesis for modelling. In the opposite, the two following sections will present situations where the function F is not increasing, hence where $(\mathcal{H})$ does not hold.

2 Hazard process and filtration enlargement

As recalled in the introduction, whereas the pricing rule in the hazard process framework is much more convenient, this approach introduces some mathematical technicalities, that impose conditions on the credit event definition. Once the reference filtration $\mathbb{F}$ has been specified, the addition of the random time enlarges the information into a filtration $\mathbb{G}$. We now interpret the filtration $\mathbb{F}$ as the default-free information, i.e., the filtration generated by default-free assets - or by assets through which the credit event can not be observed (as, for example default free zero-coupon bonds).

It is known that so that to preclude arbitrages in the default-free market, the (properly discounted) asset prices are $\mathbb{F}$-semi-martingales. As the full market is assumed to be arbitrage free, these prices must stay $\mathbb{G}$ -semi-martingales, i.e., we must have $\mathbb{F}\hookrightarrow_{(\mathcal{H}')}\mathbb{G}$. Unfortunately, $(\mathcal{H}')$-hypothesis is not satisfied in general in a progressive filtration enlargement, and some technical conditions have to be imposed to the credit event for this property to hold. Moreover, in some situations, the stronger condition that the martingales of $\mathbb{F}$ must stay $\mathbb{G}$-martingales - $\mathbb{F}\hookrightarrow_{(\mathcal{H})}\mathbb{G}$ - is wanted (for example when the reference market is complete, but also for some interesting features). The questions relative to this property can be complex, and add in general new constraints to the definition of the credit event.

In this section we first present a natural framework for $(\mathcal{H}')$-hypothesis to hold, i.e., so that the modelling be arbitrage free, then make a development on the $(\mathcal{H})$-hypothesis and finish with examples of constructions for default times based on Cox Processes idea.

[6] Under our hypotheses, since if λ is $\mathbb{F}$-adapted and satisfies $\lambda_t^{\mathbb{G}} = 1\!\!1_{t<\tau}\lambda_t^{\mathbb{F}}$, then $\lambda_t^{\mathbb{F}} = \mathbb{E}(1\!\!1_{t<\tau}\lambda_t^{\mathbb{G}}|\mathcal{F}_t)/G_{t-}$.

2.1 $(\mathcal{H}')$-hypothesis

Enlargements of filtrations have been extensively studied so far and we only recall the main formulas, that are necessary for the sequel. Refer to [23], [24], [36], [31] or [34] for proper presentations.

The property that $\mathbb{F}$-semi-martingales remain $\mathbb{G}$ -semi-martingales is often called $(\mathcal{H}')$ hypothesis. In the case of a progressive enlargement, $(\mathcal{H}')$ hypothesis is always satisfied until the default time in the following sense: if $X \in \mathcal{M}(\mathbb{F},\mathbb{P})$, the process X stopped at the credit event, i.e., the process $(X_t^\tau = X_{\tau\wedge t}, t \geq 0)$ is a $\mathbb{G}$-semi-martingale. Indeed if $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = M_t - A_t$ ($\mathbb{F}$-Doob Meyer decomposition) and if B is the $\mathbb{F}$-predictable dual projection of the $\mathbb{G}$-adapted process $(\varepsilon_u)_u$=$(\Delta X_\tau H_u)_u$ (this process is *equal to zero* under the (classical) assumption (often called (A)-condition) that τ avoids the $\mathbb{F}$-stopping times: $\mathbb{P}(\tau = T) = 0$ for all T $\mathbb{F}$-stopping time), Jeulin's formula states that

$$X_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{d\langle X, M\rangle_u + dB_u}{G_{u-}} \in \mathcal{M}(\mathbb{G},\mathbb{P}) \tag{4}$$

(see [23] for example). The situation after the credit event is more complicated, and conditions must be imposed to the random time such that $(\mathcal{H}')$ hypothesis holds.

The two more common cases, under which $(\mathcal{H}')$ hypothesis holds, are when τ is a *honest time*, or is an *initial time*. Precisely, let $X \in \mathcal{M}(\mathbb{F},\mathbb{P})$ and denote by $G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t)$ (remark $G_t = G_t^t$). We have, with above notations, the following results:

- The credit event τ is said to be a *honest time* if for any $t > 0$, the r.v. τ is equal to an $\mathcal{F}_t$-measurable random variable on $\{\tau \leq t\}$. In that case:

$$X_t - \int_0^{t\wedge\tau} \frac{d\langle X, M\rangle_u + dB_u}{G_{u-}} + 1\!\!1_{\{\tau\leq t\}} \int_\tau^t \frac{d\langle X, M\rangle_u + dB_u}{F_{u-}} \in \mathcal{M}(\mathbb{G},\mathbb{P}).$$

- The credit event τ is said to be an *initial time* if there exists a family of positive $\mathbb{F}$-martingales $(\alpha_t^u, t \geq 0)$ such that $G_t^T = \int_T^\infty \alpha_t^u \eta(du)$, where η is a finite non-negative measure on $\mathbb{R}^+$. Refer to [37] or [19] for a study of the properties of these times. In that case, we can write (see [19]):

$$X_t - \int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_u + dB_u}{G_{u-}} - \int_{t\wedge\tau}^t \left. \frac{d\langle X, \alpha^\theta\rangle_u}{\alpha_{u-}^\theta} \right|_{\theta=\tau} \in \mathcal{M}(\mathbb{G},\mathbb{P}).$$

Initial times are very appropriate for the study of a credit event (or of many credit events), which is not the case of honest times. Indeed these times necessarily belong to $\mathcal{F}_\infty$, which is not the case in general of a credit event (see example at the end of this section). Moreover, after the credit event the $\mathbb{G}$-adapted process depend in general on the credit event. This feature is impossible if the time is honest (every $\mathbb{G}$-predictable process is $\mathbb{F}$-measurable after the credit event by definition of honesty).

2.2 $(\mathcal{H})$-hypothesis

A very particular case of enlargement of filtration corresponds to the immersion property: there is immersion between $\mathbb{F}$ and $\mathbb{G}$ is any $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale ($\mathcal{M}(\mathbb{F}) \subset \mathcal{M}(\mathbb{G})$). Brémaud and Yor [6] gave a simple characterization of immersion proving its equivalence with: $\forall t > 0$, $\mathcal{F}_\infty$ is independent with $\mathcal{G}_t$ conditionally to $\mathcal{F}_t$. As proved in [19], there exists a simple characterization for immersion when the credit event is modelled by an initial time: if it avoids the $\mathbb{F}$-stopping times (for the sake of simplicity), there is equivalence between $\mathbb{F} \hookrightarrow_{(\mathcal{H})} \mathbb{G}$ and for any $u \geq 0$, the martingale α^u is constant after u. We make also the technical assumption that G does not vanish.

We have seen that within the enlarged filtration framework, the $HBPR$ allows to compute the price of defaultable claims very easily: at the opposite of $IBPR$, it does not involve the computation of the jump of any process. We shall see that if the reference filtration is immersed into the full filtration, these two formulas are very close.

If the reference filtration is immersed into the full filtration, then G has no martingale part, i.e., is a non increasing predictable process. Indeed, using the Doob-Meyer decomposition of G as $G = M - A$ (with $M_0 = 1$), by immersion $M \in \mathcal{M}(\mathbb{F}) \subset \mathcal{M}(\mathbb{G})$, and as τ is a $\mathbb{G}$-stopping time, optional sampling theorem implies that $M^\tau \in \mathcal{M}(\mathbb{G})$. It follows from (4), that $\int_0^{t\wedge\tau} d\langle M, M\rangle_u / G_{u-}$ is a predictable increasing martingale, hence is constant, equal to 0. It follows $d\langle M, M\rangle_u^\tau = 0$, hence $\langle M, M\rangle_u^\tau$ is constant and M^τ is constant equal to $M_0 = 1$. Moreover, $T = \inf\{t > 0, M_t \neq 1\}$ is an $\mathbb{F}$-stopping time and $\tau \leq T$. It follows $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) \leq \mathbb{P}(T > t|\mathcal{F}_t) = 1_{\{T>t\}}$. If G does not vanishes, $T = \infty$ and $M = 1$, which proves that G is decreasing and predictable for any $t \geq 0$ if immersion holds (remark that the fact that G is decreasing under immersion is straightforward, since under immersion $G_t = P(\tau < t|\mathcal{F}_t) = P(\tau \leq t|\mathcal{F}_\infty)$).

It follows that the hazard process $\Gamma = -\ln G$ is an increasing predictable process and

$$\Lambda_t^{\mathbb{F}} = \int_0^t \frac{dA_s}{G_{s-}} = \int_0^t \frac{dG_s}{G_{s-}} = \Gamma_t,$$

where the last equality holds if G is continuous. It follows $\Lambda_t = \Gamma_{t\wedge\tau}$. If Γ is continuous w.r.t Lebesgue measure, $\Gamma_t = \int_0^t \lambda_s^{\mathbb{F}} ds$, and $HBPR$ becomes:

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} e^{\Gamma_t} \mathbb{E}\left(X e^{-\Gamma_T} |\mathcal{F}_t\right) = 1\!\!1_{\{\tau>t\}} \mathbb{E}\left(X e^{-\int_t^T \lambda_s^{\mathbb{F}} ds} \middle| \mathcal{F}_t \right).$$

To compare the two pricing rules, we need to remark that under $(\mathcal{H})$ hypothesis, for any $\mathcal{F}_T$-measurable integrable random variable X :

$$\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t) \text{ for every } t \leq T.$$

Indeed if immersion holds these two $\mathbb{G}$-martingales have the same terminal value X. It follows

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \mathbb{E}\left(X e^{-\int_t^T \lambda_s^{\mathbb{F}} ds} \middle| \mathcal{F}_t \right) = 1\!\!1_{\{\tau>t\}} \mathbb{E}\left(X e^{-\int_t^T \lambda_s^{\mathbb{F}} ds} \middle| \mathcal{G}_t \right)$$

that has to be compared to $IBPR$:

$$\mathbb{E}(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \mathbb{E}\left(X e^{-\int_t^T \lambda_s^{\mathbb{G}} ds} \middle| \mathcal{G}_t \right) - \mathbb{E}(\Delta V_\tau 1\!\!1_{\{t<\tau\leq T\}}|\mathcal{G}_t).$$

It appears that $HBPR$ behaves like the $IBPR$ where the intensity would have been replaced by the $\mathbb{F}$-intensity and where the jump would vanish[7]. In that sense, we say that $IBPR$ and

[7] Remark that this point is just an interpretation, and that it is not true to write that the filtration enlargement allows to construct a version of the intensity under which the jump disappears. Indeed in such a framework, a true application (the intensity) of $IBPR$ leads to the expression of the jump:

$$\begin{aligned} \mathbb{E}(\Delta V_\tau 1\!\!1_{\{t<\tau\leq T\}}|\mathcal{G}_t) &= 1\!\!1_{\{t<\tau\}} e^{\Gamma_t} \left(\mathbb{E}\left(X e^{-\Lambda_{T\wedge\tau}} \middle| \mathcal{G}_t \right) - 1\!\!1_{\{\tau>t\}} \mathbb{E}\left(X e^{-\Gamma_T} \middle| \mathcal{G}_t \right) \right) \\ &= 1\!\!1_{\{t<\tau\}} e^{\Gamma_t} \mathbb{E}\left(1\!\!1_{\{\tau\leq T\}} X \left(e^{-\Gamma_\tau} - e^{-\Gamma_T} \right) \middle| \mathcal{G}_t \right). \end{aligned}$$

$HBPR$ are close in an immersed context. Remark also that in this case, if τ is the default time of a defaultable asset, the "$\mathbb{F}$-intensity" associated to τ can be interpreted as its spread over the interest rate. Indeed if D the price process of a defaultable zero-coupon bond (associated to the credit event τ),

$$D(t,T) = \mathbb{1}_{\{t<\tau\}}\mathbb{E}\left(e^{-\int_t^T \left(r_s+\lambda_s^{\mathbb{F}}\right)ds}\middle|\, \mathcal{F}_t\right) = \mathbb{1}_{\{t<\tau\}}\mathbb{E}\left(e^{-\int_t^T \left(r_s+\lambda_s^{\mathbb{F}}\right)ds}\middle|\, \mathcal{G}_t\right).$$

The most commonly used application of this framework is the Cox process construction, presented in the following section (see [27]).

Another reason why the study of immersion is important, is the following arbitrage condition. If the reference market is complete, $(\mathcal{H})$ hypothesis holds necessarily, to avoid arbitrages while using $\mathbb{G}$ -adapted strategies, as proved in [5]. Indeed[8] if $M_t = \mathbb{E}(M_T|\mathcal{F}_t)$ is a $\mathbb{F}$-martingale, the market being arbitrage free, it represents an arbitrage price (a price that does not lead to an arbitrage) of the claim M_T. The market being complete, there exists a replicating strategy, which is $\mathbb{F}$-adapted (and the price is unique, as well as the martingale measure). By hypothesis, the total Market remains arbitrage free, so there exits at least a $\mathbb{Q}^*$ equivalent martingale probability, i.e., under which the dynamics of S is a $\mathbb{G}$-martingale. The $\mathbb{F}$-Market being complete and arbitrage free, $\mathbb{Q}^*$ restricted to $\mathbb{F}$ must coincide with $\mathbb{Q}$ (it is not true in incomplete setting). An arbitrage price of the M_T claim in the total market is $\mathbb{E}^*(M_T|\mathcal{G}_t)$. As the claim is replicable (a $\mathbb{F}$-admissible strategy remains a $\mathbb{G}$ -admissible strategy), the price is unique and $M_t = \mathbb{E}^*(M_T|\mathcal{G}_t)$ is a $\mathbb{G}$-martingale.

2.3 Examples of constructions of credit event

Cox construction of the random time - under which an $\mathbb{F}$–adapted process crosses an independent trigger - allows for a very simple and intuitive method to define the credit event, leading to non $\mathcal{F}_\infty$-measurable times. It is known that $(\mathcal{H})$ hypothesis (hence $(\mathcal{H}')$ hypothesis) holds in such type of progressive enlargement (these times are in fact initial under an additional hypothesis). A slight modification of the construction leads to a violation of this property. Indeed, recall that in a progressive expansion of filtration, i.e., for $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, the assertion: $(\mathcal{H})$-hypothesis holds between $\mathbb{F}$ and $\mathbb{G}$, is equivalent to the conditionally independence of $\mathcal{F}_\infty$ and $\mathcal{H}_t$ given $\mathcal{F}_t$. It follows that $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t) = \mathbb{P}(\tau \le t|\mathcal{F}_\infty)$ is increasing. Breaking the increasing property of F implies letting $\mathcal{H}_t$ not be independent with $\mathcal{F}_\infty$ given $\mathcal{F}_t$ anymore. In Cox construction, the default is triggered as the $\mathbb{F}$ clock overtakes an independent barrier Θ. Henceforth $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are independent given $\mathcal{F}_t$, since Θ does not depend on $\mathcal{F}_\infty$. A source of noise Θ that does not belong to $\mathcal{F}_\infty$ remains necessary for the default time to get out the $\mathbb{F}$ information. Letting this trigger having a $\mathcal{F}_\infty$ non independent part will violate the $(\mathcal{H})$-hypothesis, and allow for a broader class of models.

We propose in this section two examples: the classical Cox construction, where immersion property holds and two variations where immersion property does not hold but where the random times are initial times, hence $(\mathcal{H}')$-hypothesis is satisfied. In the following examples, a filtration $\mathbb{F}$ is given, as well as an $\mathbb{F}$ -adapted process X, and one or two non negative r.vs: V which is $\mathcal{F}_\infty$ -measurable and integrable, and Θ which is independent of $\mathcal{F}_\infty$ with unit exponential law.

- In the Cox process construction, τ is given by

$$\tau = \inf\{t : X_t \ge \Theta\}.$$

[8] Assume the filtration is generated by the asset S which is martingale under the probability $\mathbb{Q}$ (the risk neutral probability with no interest rate to ease the notation).

In that case, if $\Lambda_t := \sup_{s \le t} X_s$,

$$G_t = \mathbb{P}(\Lambda_t < \Theta | \mathcal{F}_t) = \exp(-\Lambda_t) = 1 - A_t,$$

and, if Λ is continuous, the process

$$H_t - \int_0^{t \wedge \tau} \frac{dA_s}{1 - F_s} = H_t - \Lambda_{t \wedge \tau}$$

is a $\mathbb{G}$-martingale. See for example [27] and [15] for a study when Λ is not absolutely continuous. It is proved in [19] that these times are initial (with known martingale density, under the assumption that Λ is absolutely continuous), and that immersion property holds. It follows that for $X \in \mathcal{F}_T$, one has the pricing rule:

$$\mathbb{E}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}(e^{-\Gamma_T} X | \mathcal{F}_t) = 1\!\!1_{\{\tau > t\}} e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T} X | \mathcal{F}_t).$$

It has been proved by N. El Karoui [11] that when $(\mathcal{H})$ hypothesis holds and G is continuous, the default time can be constructed as in Cox process method ("canonical construction"), Γ_τ playing the rôle of the stochastic barrier Θ independent of $\mathcal{F}_\infty$. The proof is based on the fact that Γ is increasing and inversible, its continuity assuring the relation $\Gamma_{C_t} = t$ if C is its inverse process.

- One can extend this construction as follows: Introduce the random time:

  $$\tau = \inf\{t : \Lambda_t \ge \Theta V\}$$

  The variable ΘV is not independent from $\mathcal{F}_\infty$. Let us define

  $$\psi_s = \frac{\lambda_s}{V} \exp\left(- \int_0^s \frac{\lambda_u}{V} du \right)$$

  and for any pair (s,t), we set $\gamma(s,t) = \mathbb{E}\left(\psi_s | \mathcal{F}_t \right)$, so that $\gamma(s,t)$ is non-negative and, for any s, the process $(\gamma(s,t),\, t \ge 0)$ is an $\mathbb{F}$-martingale. The law of the variable τ writes $\eta([0,t]) = \int_0^t \gamma(s,0) ds = \mathbb{P}(\tau \le t)$, and for any T and t :

  $$G_t^T = \mathbb{P}(\tau > T | \mathcal{F}_t) = \int_T^\infty \alpha_t^s \eta(ds)$$

  with $\alpha_t^s = \gamma(s,t)/\gamma(s,0)$. It follows that such times are initial. Moreover, in this situation, if $t \ge s$, $\alpha_t^s \ne \alpha_s^s$ hence immersion does not hold. If $0 < V < 1$, a.s., such a time is almost surely finite. See [19] for more details.

3 The example of incomplete information

This part presents a natural situation where immersion fails to hold: the reduction of information. Starting from a framework where immersion holds, it is easy to prove that a filtration shrinking (a projection on discrete dates for example) breaks this property. A more sophisticated generalization of this simple (but powerful) remark is the incomplete information set up, illustrated with the Guo et al. [16] construction of the credit event. This set up is extensively studied in the second part of this section.

3.1 Remarks about information reduction

A very simple and natural situation under which immersion does not hold can be constructed by reducing the information (for example through a time discretization). Starting from an immersed set up ($\mathbb{F} \hookrightarrow_{(\mathcal{H})} \mathbb{G} = \mathbb{F} \vee \mathbb{H}$), a shrinking of filtration does not preserve this property in general, i.e., if $\widetilde{\mathbb{F}} \subset \mathbb{F}$ the statement $\widetilde{\mathbb{F}} \hookrightarrow_{(\mathcal{H})} \widetilde{\mathbb{G}}=\widetilde{\mathbb{F}} \vee \mathbb{H}$ does not hold in most situations.

Denote by $G_t = Z_t - A_t$ the $\mathbb{F}$-Doob-Meyer decomposition of the $\mathbb{F}$-submartingale $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ (assumed to be continuous) and assume the increasing part satisfies $dA_t = a_t dt$. The $\mathbb{F}$–intensity rate writes $\lambda_s = a_s/G_s$. Let $\widetilde{A}$ be the $\widetilde{\mathbb{F}}$ -submartingale $\widetilde{A}_t = \mathbb{E}(A_t|\widetilde{\mathcal{F}}_t)$, and denote $\widetilde{z}_t + \widetilde{\alpha}_t$ the Doob-Meyer decomposition of this $\widetilde{\mathbb{F}}$ -submartingale (with $\widetilde{z} \in \mathcal{M}\left(\widetilde{\mathbb{F}}\right)$ and $\widetilde{\alpha} \in \mathcal{P}\left(\widetilde{\mathbb{F}}\right)$ increasing). Then, setting $\widetilde{Z}_t = \mathbb{E}\left(Z_t|\widetilde{\mathcal{F}}_t\right) \in \mathcal{M}\left(\widetilde{\mathbb{F}}\right)$, the $\widetilde{\mathbb{F}}$-Doob-Meyer decomposition of $\widetilde{G}$ is

$$\widetilde{G}_t = \mathbb{P}\left(\tau > t|\widetilde{\mathcal{F}}_t\right) = \mathbb{E}\left(G_t|\widetilde{\mathcal{F}}_t\right) = \widetilde{Z}_t - \widetilde{z}_t - \widetilde{\alpha}_t \equiv \widetilde{X}_t - \widetilde{\alpha}_t,$$

where $\widetilde{X}_t \in \mathcal{M}\left(\widetilde{\mathbb{F}}\right)$ and $\widetilde{\alpha}_t = \int_0^t \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)ds$ is a $\widetilde{\mathbb{F}}$-predictable increasing process[9]:

Starting from a simple framework where immersion holds (with $G = 1 - A$), it is straightforward to build an example where the hazard process is not increasing anymore: the reduction of information makes appear a martingale part in the decomposition of the hazard process ($\widetilde{z}_t$, with above notations).

Proposition 1 *(i) If $\mathbb{P}(\tau > t|\mathcal{F}_t) = G_t = Z_t - \int_0^t a_s ds$ with $Z \in \mathcal{M}(\mathbb{F})$, and if G is continuous, the $\mathbb{F}$–intensity rate of τ writes $\lambda_t = a_t/G_t$.*

(ii) If $\widetilde{\mathbb{F}} \subset \mathbb{F}$, $\mathbb{P}\left(\tau > t|\widetilde{\mathcal{F}}_t\right) = \widetilde{G}_t = \widetilde{X}_t - \int_0^t \widetilde{a}_s ds$ with $\widetilde{X} \in \mathcal{M}\left(\widetilde{\mathbb{F}}\right)$ and $\widetilde{a}_s = \mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)$. Moreover: $H_t - \int_0^{t\wedge\tau} \widetilde{a}_s/\widetilde{G}_s ds$ is a $\widetilde{\mathbb{G}}$ martingale, and the $\widetilde{\mathbb{F}}$-intensity rate of τ writes $\widetilde{\lambda}_t = \widetilde{a}_t/\widetilde{G}_t = \mathbb{E}\left(\lambda_t G_t|\widetilde{\mathcal{F}}_t\right)/\mathbb{E}\left(G_t|\widetilde{\mathcal{F}}_t\right)$.

Remark that the intensity in the reduced model is not the optional projection of the intensity (i.e. $\widetilde{\lambda}_t \neq \mathbb{E}(\lambda_s|\widetilde{\mathcal{F}}_s) = \mathbb{E}(a_s/G_s|\widetilde{\mathcal{F}}_s)$) but $\mathbb{E}(a_s|\widetilde{\mathcal{F}}_s)/\widetilde{G}_s$: *it underlines the fact that computing the intensity in the reduced model necessitates the knowledge of the Doob-Meyer decomposition of* $\widetilde{F}_t$. Such a result could not hold even in simple situations. If we consider the example of a trivial sub-filtration $\widetilde{\mathbb{F}}$, the $\widetilde{\mathbb{F}}$-intensity rate writes $\widetilde{\lambda}_s = f(s)/G(s)$ (with $G(s) = \mathbb{P}(\tau > s) = \int_s^\infty f(u)du$), and the equality $\widetilde{\lambda}_s = \mathbb{E}(\lambda_s|\widetilde{\mathcal{F}}_s) = \mathbb{E}(\lambda_s)$ would imply $G(t) = \exp - \int_0^t \mathbb{E}(\lambda_s)ds$ (solving the ODE). By assumption

$$\mathbb{P}(\tau > t\,|\mathcal{F}_t) = \exp - \int_0^t \lambda_s ds,$$

and it follows by taking the expectation that $G(t) = \mathbb{E}\left(\exp - \int_0^t \lambda_s ds\right)$ hence:

$$\mathbb{E}\left(\exp - \int_0^t \lambda_s ds\right) = \exp - \int_0^t \mathbb{E}(\lambda_s)ds \text{ for any } t \geq 0.$$

[9] This result is very easy to check : if $N_t = \widetilde{A}_t - \int_0^t \mathbb{E}(a_u|\widetilde{\mathcal{F}}_u)du$ and $c_s \in \widetilde{\mathcal{F}}_s$,

$$\mathbb{E}\left(c_s\left(N_t - N_s\right)\right) = \int_s^t \mathbb{E}\left(\mathbb{E}(c_s a_u|\widetilde{\mathcal{F}}_t) - \mathbb{E}(c_s a_u|\widetilde{\mathcal{F}}_u)\right)du = \int_s^t \mathbb{E}(c_s a_u) - \mathbb{E}(c_s a_u)du = 0.$$

The exponential concave function being non affine, the equality in Jensen's inequality can only be achieve if $\lambda_s = \mathbb{E}(\lambda_s)$ is deterministic. The relation $\widetilde{\lambda}_t = \mathbb{E}(\lambda_s|\widetilde{\mathcal{F}}_s)$ would necessary imply λ deterministic.

It is worth mentioning that if the random time τ is an $\mathbb{F}$-initial time, it remains an $\widetilde{\mathbb{F}}$-initial time. Indeed if $G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \int_T^\infty \alpha_t^u du$, with for any $u \geq 0$, $\alpha_\cdot^u \in \mathcal{M}(\mathbb{F})$

$$\widetilde{G}_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t|\widetilde{\mathcal{F}_t}) = \int_T^\infty \mathbb{E}\left(\alpha_t^u \middle| \widetilde{\mathcal{F}}_t\right) du = \int_T^\infty \widetilde{\alpha}_t^u du$$

and for $u \geq 0$, $\widetilde{\alpha}_\cdot^u \in \mathcal{M}\left(\widetilde{\mathbb{F}}\right)$. If $\mathbb{F} \hookrightarrow_{(\mathcal{H})} \mathbb{G}$, $\alpha_{u\wedge t}^u = \alpha_u^u$ but $\widetilde{\alpha}_{u\wedge t}^u$ may not be equal to $\widetilde{\alpha}_u^u$ in general and immersion not hold between $\widetilde{\mathbb{F}}$ and $\widetilde{\mathbb{G}}$ (see [19]).

3.2 Presentation of the model of delayed information

In Guo et al. [16], the authors suggest to start from a structural model and delay the information flow. From a structural approach - where the default time is the predictable hitting time by a diffusion process of a constant trigger (and has therefore no intensity) - they derive a reduced form modelling by altering the initial information. In this delayed information framework, the default time has yet an intensity. Guo et al. derive explicit analytic connections between default intensities and the density functions of the corresponding first passage times for general continuous time Markov models.

Precisely, let us consider a continuous Markov process X defined on a space $(\Omega, \mathcal{A}, (\mathbb{P}_x)_{x\in\mathbb{R}}, \theta)$ where for each x, the probability $\mathbb{P}_x$ satisfies $\mathbb{P}_x(X_0 = x) = 1$ and θ is a translation on Ω (i.e., $X_s \circ \theta_t = X_{s+t}$)[10]. We denote by $\mathbb{F}$ the natural augmentation of the filtration generated by X. The authors start from a structural model where τ_b is an $\mathbb{F}$-predictable stopping time defined by

$$\tau_b = \inf\{t > 0, X_t \leq b\},$$

for a fixed $b \in \mathbb{R}$. For the sake of notational simplicity we shall note $\tau = \tau_b$. If $\delta > 0$, introduce, for $t > \delta$ the σ -algebra $\widetilde{\mathcal{F}}_t = \mathcal{F}_{t-\delta} \subset \mathcal{F}_t$, and for $0 < t < \delta$ $\widetilde{\mathcal{F}}_t$ equal to the trivial filtration. We set $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$.

Remark that since τ is an $\mathbb{F}$-stopping time, it can not be $\mathbb{F}$-initial. It follows it would not be possible to try and use last section's result to establish any $\widetilde{\mathbb{F}}$-initial property of τ. Anyways, it is clear that this time is not $\widetilde{\mathbb{F}}$-initial, since for any $T \leq t-\delta$, $\widetilde{G}_t^T = 1_{\{\tau>T\}}$ hence is not absolutely continuous w.r.t. a deterministic measure on $\mathbb{R}^+$. However, the $\widetilde{\mathbb{F}}$-semi-martingales will remain at all time $\widetilde{\mathbb{G}}$-semi-martingales. Indeed such a construction of the credit event is honest, since default, i.e., on $\{\tau \leq t\}$, the time $\tau - \delta$ can be expressed in terms of $\widetilde{\mathcal{F}}_t$-measurable elements. On $\{\tau \leq t\}$, $1_{\{\eta \leq t-\delta\}} = 1$, with $\eta = \inf\{t > 0, Z_t \leq b\}$, hence $1_{\{\eta\leq t-\delta\}}\eta$ is $\widetilde{\mathcal{F}}_t$-measurable. Finally on $\{\tau \leq t\}$ $\tau = \tau_t$ with $\tau_t = 1_{\{\eta\leq t-\delta\}}\eta + \delta \in \widetilde{\mathcal{F}}_t$, hence τ is $\widetilde{\mathbb{F}}$-honest, and $(\mathcal{H}')$-hypothesis holds between $\widetilde{\mathbb{F}}$ and $\widetilde{\mathbb{G}}$.

Guo et al. study this model via a pure intensity approach. As presented in the first part of this survey, the classical method to derive the $\widetilde{\mathbb{G}}$-intensity is based on the application of Aven's lemma (or approximated Laplacians method):

$$\lambda_t = \lim_{h\to 0} \frac{1}{h}\mathbb{P}_x\left(t < \tau \leq t+h \middle| \widetilde{\mathcal{G}}_t\right) = \mathbb{1}_{\{t<\tau\}} \frac{f(X_{t-\delta}, b, \delta)}{\mathbb{P}_{X_{t-\delta}}(\delta < \tau)}$$

[10] The existence of a translation is a classical technical assumption in the study of Markov processes

where the second equality comes from the Markov property of X and from the introduction of the continuous density function of τ, $f(x,b,t) = \mathbb{P}_x(\tau_b \in]t, t+dt])/dt$ (see [16]).

The pricing rule $IBPR$ for a terminal claim writes

$$\mathbb{E}(X\mathbb{1}_{T<\tau}|\widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{t<\tau\}}\left(V_t - \mathbb{E}(\Delta V_\tau \mathbb{1}_{\{\tau\leq T\}}|\widetilde{\mathcal{G}}_t)\right)$$

and involves the computation of the jump at τ of the process

$$V_t = \exp\int_0^{t\wedge\tau}\frac{f(X_{s-\delta},b,\delta)}{\mathbb{P}_{X_{s-\delta}}(\delta<\tau)}ds\,\mathbb{E}\left(X\exp\int_0^{T\wedge\tau}\frac{f(X_{s-\delta},b,\delta)}{\mathbb{P}_{X_{s-\delta}}(\delta<\tau)}ds\,\middle|\,\widetilde{\mathcal{G}}_t\right).$$

This jump has no reason to be equal to zero, and the $IBPR$ is not tractable. It is therefore more convenient to compute the $\widetilde{\mathbb{F}}$-hazard process and to use the $HBPR$.

3.3 Pricing defaultable claims

Let us start with a short remark about the translation. Recall that if X is a continuous $\mathbb{F}$-martingale, the process Z defined as $Z_t = X_{t-\delta}\mathbb{1}_{t>\delta} + X_0\mathbb{1}_{t<\delta}$ is a continuous $\widetilde{\mathbb{F}}$-martingale[11]. The bracket of this martingale writes $\langle Z\rangle_t = \langle X\rangle_{t-\delta}\mathbb{1}_{t>\delta}$ (from a direct computation of the Riemann sum for instance).

Computation of the $\widetilde{\mathbb{F}}$-conditional survival process (or equivalently the $\widetilde{\mathbb{F}}$-hazard process). As $\tau_b = \inf\{t>0, X_t\leq b\}$, for $t>\delta$,

$$\begin{aligned}\widetilde{G}_t &= \mathbb{P}_x(\tau_b>t|\widetilde{\mathcal{F}}_t) = \mathbb{P}_x\left(\inf_{s\leq t}X_s>b\,\middle|\,\widetilde{\mathcal{F}}_t\right) = \mathbb{1}_{\inf_{s\leq t-\delta}X_s>b}\mathbb{P}_x\left(\inf_{t-\delta<s\leq t}X_s>b\,\middle|\,\widetilde{\mathcal{F}}_t\right)\\ &= \mathbb{1}_{\inf_{s\leq t-\delta}X_s>b}\mathbb{P}_{X_{t-\delta}}\left(\inf_{s\leq\delta}X_s>b\right) = \mathbb{1}_{\inf_{s\leq t-\delta}X_s>b}\Phi(X_{t-\delta},\delta,b) = D_t\Phi(Z_t,\delta,b)\end{aligned}$$

where $D_t = \mathbb{1}_{\inf_{s\leq t-\delta}X_s>b}$, $Z_t = X_{t-\delta}$ and $\Phi(x,u,y) = \mathbb{P}_x(\inf_{s\leq u}X_s>y)$. The process $\widetilde{G}$ is not decreasing (and does not have finite variation if X has non finite variation), hence $(\mathcal{H})$ hypothesis does not hold in this framework (the knowledge of the intensity is not enough to compute the value of defaultable claims, except if one is able to compute the needed jump). If $t\leq\delta$, $\widetilde{G}_t = \mathbb{P}_x(\tau_b>t) = \Phi(x,t,b)$.

Dynamics of $\widetilde{G}$. For $t>\delta$, integration by parts formula leads to

$$d\widetilde{G}_t = D_t d\Phi(Z_t,\delta,b) + \Phi(Z_t,\delta,b)dD_t$$

since the process $\Phi(Z,\delta,b)$ is continuous and D has finite variation. Applying Itô's formula to $\Phi(Z_t,\delta,b)$, we can write

$$d\widetilde{G}_t = D_t\partial_1\Phi(Z_t,\delta,b)dZ_t + \frac{1}{2}D_t\partial_{1,1}\Phi(Z_t,\delta,b)d\langle Z\rangle_t + \Phi(Z_t,\delta,b)dD_t.$$

[11] indeed (i) if $t\leq\delta, \widetilde{\mathcal{F}}_t$ is trivial: if $T\leq\delta, Z_T = Z_t = X_0$ hence, $E(Z_T|\widetilde{\mathcal{F}}_t) = X_0 = Z_t$ and if $\delta<T$, $E(Z_T|\widetilde{\mathcal{F}}_t) = E(Z_T) = E(Z_{T-\delta}) = X_0$,
(ii) if $t>\delta$, $E(Z_T|\widetilde{\mathcal{F}}_t) = E(X_{T-\delta}|\mathcal{F}_{t-\delta}) = W_{t-\delta} = Z_t$.

Because D only jumps at times t such that $Z_t = b$ and since $\Phi(b, \delta, b) = 0$, the last term of the right- hand side of last equality is identically null. Therefore

$$d\widetilde{G}_t = D_t \partial_1 \Phi(Z_t, \delta, b) dZ_t + \frac{1}{2} D_t \partial_{1,1} \Phi(Z_t, \delta, b) d \langle Z \rangle_t .$$

It follows (from the decomposition of Z) the decomposition of the (special) $\widetilde{\mathbb{F}}$-semi-martingale $\widetilde{G}$, since $\langle Z \rangle$ is $\widetilde{\mathbb{F}}$-predictable. We check that the martingale part M of the survival process is not constant.

Pricing contingent claim. The pricing through $HBPR$ is now easy and standard. The price of a defaultable contingent claim with payoff $f(X_T)\mathbb{1}_{T<\tau}$ is

$$\mathbb{E}(f(X_T)\mathbb{1}_{T<\tau} | \widetilde{\mathcal{G}}_t) = (1 - H_t) \frac{\mathbb{1}_{\{\widetilde{G}_t > 0\}}}{\widetilde{G}_t} \mathbb{E}\left(\widetilde{G}_T f(X_T) | \widetilde{\mathcal{F}}_t\right).$$

Indeed, this formula is the classical one when the survival process does not reach zero[12], and when the $\widetilde{G}_t = 0$, it is classical that $\tau \leq t$ (see for example theorem 13 in [34]) and the two members are equal to zero (remark $\mathbb{E}(f(X_T)\mathbb{1}_{T<\tau} | \widetilde{\mathcal{G}}_t) = \mathbb{1}_{t<\tau} \mathbb{E}(f(X_T)\mathbb{1}_{T<\tau} | \widetilde{\mathcal{G}}_t)$). The conditional expectation can be computed using the Markov property of Z :

$$\mathbb{E}\left(\widetilde{G}_T f(X_T) | \widetilde{\mathcal{F}}_t\right) = V(t, T, Z_t)$$

where V, assumed to be smooth, satisfies a PDE.

Recovering the $\widetilde{\mathbb{F}}$-intensity. As seen previously, it can be recovered from the hazard process. If the Markov process X follows the homogeneous diffusion $X_t = x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s$, we can write with $\beta_s = W_s - W_\delta$, $\widetilde{\mu}(s,x) = \mathbb{1}_{s>\delta}\mu(x)$ and $\widetilde{\sigma}(s,x) = \mathbb{1}_{s>\delta}\sigma(x)$:

$$Z_t = x + \mathbb{1}_{t>\delta} \int_\delta^t \mu(Z_s)ds + \mathbb{1}_{t>\delta} \int_\delta^t \sigma(Z_s)d\beta_s = x + \int_0^t \widetilde{\mu}(s, Z_s)ds + \int_\delta^t \widetilde{\sigma}(s, Z_s)d\beta_s$$

From previous calculations we have for $s \geq \delta$,

$$d\widetilde{A}_s = -D_s \left(\partial_1 \Phi(Z_s, \delta, b)\mu(Z_s) + \frac{1}{2}\partial_{11}\Phi(Z_s, \delta, b)\sigma^2(Z_s)\right) ds,$$

and the $\mathbb{F}$-intensity (since $\widetilde{G}_{s-} = D_s \Phi(Z_s, \delta, b)$) is, for $s > \delta$:

$$\widetilde{\lambda}_s = -D_s \frac{\partial_1 \Phi(Z_s, \delta, b)\mu(Z_s) + \partial_{11}\Phi(Z_s, \delta, b)\sigma^2(Z_s)/2}{\Phi(Z_s, \delta, b)}.$$

We could also use a direct computation, by the approximated Laplacians:

$$\begin{aligned} d\widetilde{A}_s &= \lim_{h\to 0} \frac{1}{h}\mathbb{E}\left(\widetilde{F}_{s+h} - \widetilde{F}_s \middle| \widetilde{\mathcal{F}}_s\right) ds = -\lim_{h\to 0} \frac{1}{h}\mathbb{P}_x\left(s < \tau \leq s+h \middle| \widetilde{\mathcal{F}}_s\right) ds \\ &= -\lim_{h\to 0} \frac{1}{h} D_s \mathbb{P}_{Z_s}(\delta < \tau \leq \delta + h)\, ds = -D_s f(Z_s, \delta, b) ds. \end{aligned}$$

It follows $\widetilde{\lambda}_s = -D_s f(Z_s, \delta, b)/\Phi(Z_s, \delta, b)$, and we retrieve $\lambda_s = \mathbb{1}_{\{t<\tau\}}\widetilde{\lambda}_s$ the intensity computed by Guo et al.

[12] The proof of $\mathbb{1}_{\{\widetilde{G}_t>0\}}\mathbb{E}(f(X_T)\mathbb{1}_{T<\tau}|\widetilde{\mathcal{G}}_t) = \mathbb{1}_{\{\widetilde{G}_t>0\}}(1-H_t)\mathbb{E}\left(\widetilde{G}_T f(X_T)|\widetilde{\mathcal{F}}_t\right)/\widetilde{G}_t$ is the same as without the indicator.

3.4 Examples and remarks about completeness

This last subsection is dedicated to examples and remarks about the previous model of incomplete information. The first point mentioned is the difficulty to compute the function Φ (or function f) in that model, hence the hazard process (and the intensity). These functions depend on the properties of process X (basically on the laws of the hitting times). Recall

$$f(x,u,y) = -\partial_2 \Phi(x,u,y).$$

Examples. Let us present simple examples, where computations are simple:

- In the case where $X_t = x + B_t$ is a Brownian Motion, $\Phi(x,\delta,b)$ writes $\Phi(0,\delta,b-x)$ and

  $$\Phi(0,u,y) = \mathbb{P}_0\left(\inf_{s\le u} B_s \ge y\right) = \mathbb{P}_0\left(\sup_{s\le u} B_s \le -y\right) = \mathbb{P}_0(|B_u| \le -y).$$

  Hence

  $$\Phi(0,u,y) = \mathcal{N}\left(-\frac{y}{\sqrt{u}}\right) - \mathcal{N}\left(\frac{y}{\sqrt{u}}\right).$$

  Moreover,

  $$f(x,b,\delta) = -\partial_2 \Phi(0,\delta,b-x) = \frac{x-b}{\delta\sqrt{2\pi\delta}} \exp -\frac{(x-b)^2}{2\delta}.$$

- In the case where $X_t = x + \sigma B_t + \mu t$, $\Phi(x,\delta,b)$ writes $\Phi(0,\delta,b-x)$

  $$\Phi(0,u,y) = \mathbb{P}_0\left(\inf_{s\le u} \sigma B_s + \mu s \ge y\right) = \mathbb{P}_0\left(\inf_{s\le u} B_s + \frac{\mu}{\sigma} s \ge \frac{y}{\sigma}\right)$$

  Hence

  $$\Phi(0,u,y) = \mathcal{N}\left(\frac{u\mu - y}{\sigma\sqrt{u}}\right) - \exp\frac{2\mu y}{\sigma^2}\mathcal{N}\left(\frac{u\mu + y}{\sigma\sqrt{u}}\right)$$

- In the case where $X_t = x\exp\{\sigma B_t + \mu t\}$ is a Geometric Brownian Motion

  $$\Phi(x,u,b) = \mathbb{P}_x(\inf_{s\le u} X_s \ge b) = \mathbb{P}_0(\inf_{s\le u} \sigma B_s + \mu s \ge \ln(b/x))$$

  Hence

  $$\Phi(x,u,b) = \mathcal{N}\left(\frac{u\mu - \ln b/x}{\sigma\sqrt{u}}\right) - \exp\frac{2\mu\ln(b/x)}{\sigma^2}\mathcal{N}\left(\frac{u\mu + \ln b/x}{\sigma\sqrt{u}}\right).$$

  Moreover

  $$f(x,b,\delta) = -\partial_2 \Phi(x,\delta,b) = -\frac{\ln b/x}{\delta\sigma^2\sqrt{2\pi\delta}} \exp -\frac{(\mu\delta - \ln b/x)^2}{2\delta\sigma^2}$$

In the general case, it is difficult to obtain the form of the function Φ.

Non completeness of the market. The second point worth to mention in this presentation of models based on delayed information, is the incompleteness of the market. Intuitively, if the initial market (with savings account and stock with price process with dynamics X) is complete, it is possible in that structural approach to hedge default free and defaultable contingent claim. While working in the reduced filtration, it is still possible to hedge contingent claim with $\widetilde{\mathbb{F}}$ adapted strategies, for payoffs like $\widetilde{K}\mathbb{1}_{T-\delta<\tau}$ where $\widetilde{K} \in \mathcal{F}_{T-\delta}$, but neither for $K \in \mathcal{F}_T$, nor $K\mathbb{1}_{T<\tau}$. Even adding a defaultable zero coupon does not complete the market. As shown below, the best hedge in a mean variance sense in this situation is a translated (or projected) delta hedging, which justifies that the previous computations of expectations for pricing are relevant. Note that every $\widetilde{\mathbb{F}}$-adapted process k is constant on $[0,\delta]$, and there exists an $\mathbb{F}$-adapted process K such that $k_t = K_{t-\delta}$ for $t \geq \delta$ and $k_t = K_0$ before δ.

The full market is arbitrage free and we denote by $\mathbb{Q}$ the martingale measure associated to the numéraire savings account (equal to 1, since there is no interest rate). Assume that the dynamics of the discounted price (equal to the price since there is no interest rate) under this probability (the risk neutral probability) are $dX/X = \sigma dW$. The portfolio of the investor is composed by χ^0 units of savings account and χ units of shares X, the strategy (χ^0, χ) being $\widetilde{\mathbb{F}}$-adapted. The portfolio is assumed to be self-financed: the value V of the portfolio, defined as $V_t = \chi_t^0 + \chi_t X_t$ satisfies $dV_t = \chi_t dX_t$. Assume the strategy of the investor consists in minimizing the terminal variance of its portfolio, i.e., denoting by $\mathcal{A}^T_{\widetilde{\mathbb{F}}}$ the set of admissible strategies on $[0,T]$:

$$J = \min_{\chi \in \mathcal{A}(\widetilde{\mathbb{F}},[0,T])} \mathbb{E}\left(V_T - Y\right)^2 .$$

Since Y is $\mathcal{F}_T$-measurable, the predictable representation theorem for a Brownian filtration leads to $Y = \mathbb{E}(Y) + \int_0^T y_s dX_s = \mathbb{E}(Y) + \int_0^T y_s \sigma_s X_s dW_s$ and

$$\begin{aligned}
J &= \min_{\chi \in \mathcal{A}^T_{\widetilde{\mathbb{F}}}} \mathbb{E}\left(V_0 + \int_0^T \chi_s dX_s - \mathbb{E}(Y) - \int_0^T y_s \sigma_s X_s dW_s\right)^2 \\
&= \min_{\chi \in \mathcal{A}^T_{\widetilde{\mathbb{F}}}} \mathbb{E}\left(V_0 - E(Y) + \int_\delta^T \chi_s dX_{s-} - \int_0^{T-\delta} y_s \sigma_s X_s dW_s - \int_{T-\delta}^T y_s \sigma_s X_s dW_s\right)^2 \\
&= \min_{\varphi \in \mathcal{A}^T_{\mathbb{F}}} \mathbb{E}\left(V_0 - E(Y) + \int_0^{T-\delta} \varphi_s \sigma_s X_s dW_s - \int_0^{T-\delta} y_s \sigma_s X_s dW_s - \int_{T-\delta}^T y_s \sigma_s X_s dW_s\right)^2 \\
&= \min_{K \in \mathcal{A}^T_{\mathbb{F}}} \mathbb{E}\left(V_0 - E(Y)\right)^2 + \mathbb{E}\left(\int_0^{T-\delta} \left(\varphi_s - y_s\right) \sigma_s X_s dW_s\right)^2 + \mathbb{E}\left(\int_{T-\delta}^T y_s \sigma_s X_s dW_s\right)^2
\end{aligned}$$

and the solution is $V_0 = \mathbb{E}(Y)$, $\chi_t = \varphi_{t-\delta} = y_{t-\delta}$ on $[\delta, T]$, and no constraint on χ_t during $[0,\delta]$, hence 0 since the variation on the asset is equal to zero (premium is invested on saving account). The replication error of this strategy is given by the last term of the right hand member, i.e.,

$$\mathbb{E}\left(\int_{T-\delta}^T y_s \sigma_s X_s dW_s\right)^2$$

which is as expected the expectation of the non-hedgeable part.

4 Pricing defaultable claims

We present in this section the pricing of two very simple products to illustrate the previous discussions: the defaultable zero coupon (with no recovery to make the example clearer), and the credit default swap. For each product, we present the computations in two different cases, both in the filtration's enlargement set-up: when $(\mathcal{H})$ hypothesis holds (under the example of a Cox-type construction, when the intensity follows a positive diffusion) and when it does not (under the example of the delayed information). In both sections, we do not address the question of the definition of the risk neutral probability, considered as given (see [3] for example, or [20]). For the sake of simplicity, we take null interest rates.

4.1 Defaultable zero coupon

In an arbitrage free model where the full information of the traders is denoted by $\mathbb{G}$, the price of the defaultable zero coupon, i.e., a product paying 1 at T if default did not occur is given by:

$$D(t,T) = \mathbb{E}(\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} e^{\Gamma_t} \mathbb{E}(e^{-\Gamma_T}|\mathcal{F}_t).$$

under a risk neutral probability $\mathbb{P}$, where Γ denotes the hazard process.

Example with immersion. In this case, if $\Gamma_t = \int_0^t \lambda_s ds$ (λ denotes the $\mathbb{F}$-intensity and $\lambda 1_{\{\tau>.\}}$ the intensity), the pricing writes:

$$D(t,T) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}(e^{\Gamma_t - \Gamma_T}|\mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}\left(e^{-\int_t^T \lambda_s ds} \middle| \mathcal{F}_t \right)$$

As seen previously, the prototype of construction satisfying $(\mathcal{H})$ hypothesis is Cox processes framework. In such a situation, the intensity can be interpreted as the spread of the obligation (replace λ_s by $\lambda_s + r_s$ if the interest rate is non null). Therefore, every class of dynamics on the short rate that leads to closed formulas for the zero coupon - $B(t,T) = E\left(\exp - \int_t^T r_s ds \middle| \mathcal{F}_t\right)$ - can be chosen for the dynamics of the intensity, and lead to closed formulas for the defaultable zero coupon. For example, if the $\mathbb{F}$-intensity follow a CIR (here B is an $\mathbb{F}$-Brownian motion):

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t} dB_t, \text{ with } 2\kappa\theta > \sigma^2, \tag{5}$$

then it is classical that (taking the formula of the zero coupon, and replacing the short rate by the intensity):

$$D(t,T) = \mathbb{1}_{\{t<\tau\}} \mathbb{E}\left(\exp - \int_t^T \lambda_s ds \middle| \mathcal{F}_t \right) = \mathbb{1}_{\{t<\tau\}} \varphi(t,T,\lambda_t)$$

with $\varphi(t,T,x) = \Phi(t,T)e^{-\Psi(t,T).x}$, and,

$$\Phi(t,T) = \left(\frac{2\eta \exp(\eta+\kappa)(T-t)/2}{2\eta + (\eta+\kappa)(\exp\eta(T-t)-1)} \right)^\mu \text{ and } \Psi(t,T) = \frac{2(\exp\eta(T-t)-1)}{2\eta + (\eta+\kappa)(\exp\eta(T-t)-1)}$$

where $\eta = \sqrt{\kappa^2 + 2\sigma^2}$, and $\mu = 2\lambda\theta/\sigma^2$:

Example without immersion. The analogy with interest rates does not hold anymore in a set up where immersion does not hold, since the Hazard process is not increasing anymore (there exists a martingale part). It follows the $\mathbb{F}$-intensity can not be directly interpreted as a spread. The value of the Defaultable Zero Coupon writes (the indicator stands for the case G could reach zero, as in Guo et al. example):

$$D(t,T) = \mathbb{1}_{\{\tau>t\}} \mathbb{1}_{\{G_t>0\}} \mathbb{E}(e^{\Gamma_t - \Gamma_T}|\mathcal{F}_t).$$

To provide an example, we consider the model of Guo et al. presented in the previous section, in which immersion does not hold. With the notations in force in that section (recall that $\mathcal{F}_t$ is replaced by $\widetilde{\mathcal{F}}_t$, $\mathcal{G}_t$ by $\widetilde{\mathcal{G}}_t$ and $D_t = \mathbb{1}_{\{\inf_{s\le t-\delta} X_s > b\}}$) the hazard process writes (beware that this process may be equal to $+\infty$):

$$\Gamma_t = -\ln\left(D_t \Phi(Z_t, \delta, b)\right).$$

It follows (from $\mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\widetilde{G}_t>0\}} = \mathbb{1}_{\{\tau>t\}}$):

$$D(t,T) = \mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\widetilde{G}_t>0\}}\mathbb{E}\left(e^{\Gamma_t-\Gamma_T}\left|\widetilde{\mathcal{F}}_t\right.\right) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left(D_T\Phi(Z_T)\left|\widetilde{\mathcal{F}}_t\right.\right)}{\Phi(Z_t)}$$

with to ease the notation $\Phi(Z_t) = \Phi(Z_t, \delta, b)$. As seen in the previous examples, the function Φ is known for suitable choices of the diffusion process X, and,

$$\mathbb{E}\left(D_T\Phi(Z_T)\left|\widetilde{\mathcal{F}}_t\right.\right) = D_t\mathbb{E}\left(\mathbb{1}_{\{\inf_{t-\delta<s\le T-\delta} X_s>b\}}\Phi(X_{T-\delta})\right|\mathcal{F}_{t-\delta}\Big) = D_t\mathbb{E}_{X_{t-\delta}}\left(D_{T-t}\Phi(X_{T-t})\right)$$

where the second equality comes from Markov property. It follows that the price of the defaultable zero coupon writes:

$$D(t,T) = D_{t+\delta}\frac{\mathbb{E}_{X_{t-\delta}}(D_{T-t}\Phi(X_{T-t}))}{\Phi(X_{t-\delta})}.$$

For example, if X is a Brownian motion $x+W$, then $\Phi(u,x) = \mathcal{N}(-x/\sqrt{u}) - \mathcal{N}(x/\sqrt{u})$, and $\mathbb{E}_{X_{t-\delta}}\left(D_{T-t}\Phi(X_{T-t})\right) = \Psi(T-t, X_{t-\delta})$, where:

$$\begin{aligned}\Psi(u,x) &= \mathbb{E}_x(\mathbb{1}_{\inf_{s\le u} X_s>b}\Phi(X_u,\delta,b)) = \mathbb{E}_0(\mathbb{1}_{\inf_{s\le u} W_s>b-x}\Phi(W_u+x,\delta,b)) \\ &= \int_0^{x-b}\int_{-\infty}^{y}\frac{2(2y-v)\Phi(v+x,\delta,b)}{\sqrt{2\pi u^3}}\exp-\frac{(2y-v)^2}{2u}dvdy.\end{aligned}$$

4.2 Credit default swap

A credit default swap is a contract in which the holder buys a protection against the default of an asset. Precisely, if a maturity T, a fee rate function $\kappa(t)$ and a recovery function $\delta(t)$ are given, the CDS of characteristics (T,κ,δ) is the contract in which the buyer of protection pays a fee at a rate κ up to default time (or to maturity if default did not happen) and receives at default time, the amount $\delta(\tau)$ from the protection seller. The price of the CDS at time t is given by the difference of the value of the two legs:

$$CDS(t,T) = Prot_t - Prem_t \equiv \mathbb{E}(\delta(\tau)\mathbb{1}_{t<\tau<T}|\mathcal{G}_t) - \mathbb{E}\left(\mathbb{1}_{t<\tau}\int_t^{\tau\wedge T}\kappa_s ds\middle|\mathcal{G}_t\right).$$

Every leg write (the values are null after default):

$$\begin{aligned}Prot_t &= \mathbb{1}_{t<\tau}\mathbb{E}\left(\int_t^T\delta(s)dH_s\middle|\mathcal{G}_t\right) = \mathbb{1}_{t<\tau}e^{\Gamma_t}\mathbb{E}\left(\int_t^T\delta(s)dA_s\middle|\mathcal{F}_t\right) \text{ and,} \\ Prem_t &= \mathbb{1}_{t<\tau}\mathbb{E}\left(\int_t^T(1-H_s)\kappa_s ds\middle|\mathcal{G}_t\right) = \mathbb{1}_{t<\tau}e^{\Gamma_t}\mathbb{E}\left(\int_t^T\kappa_s e^{-\Gamma_s}ds\middle|\mathcal{F}_t\right),\end{aligned}$$

if $G = M - A$ and recall $dH - (1-H)\,dA/G \in \mathcal{M}\,(\mathbb{G},\mathbb{P})$. Finally the price of the CDS writes :

$$CDS(t,T) = \mathbb{1}_{t<\tau}e^{\Gamma_t}\mathbb{E}\left(\int_t^T\left(\delta(s)dA_s - \kappa_s e^{-\Gamma_s}ds\right)\middle|\mathcal{F}_t\right).$$

Example with immersion. If the hazard process writes $\Gamma_t = \int_0^t\lambda_s ds$, (where λ is the $\mathbb{F}$-intensity), the formula becomes:

$$CDS(t,T) = \mathbb{1}_{t<\tau}\mathbb{E}\left(\int_t^T ds\,(\delta(s)\lambda_s - \kappa_s)\exp - \int_t^s \lambda_u du \middle| \mathcal{F}_t\right),$$

and for δ and κ constants (using the fact that Γ is increasing):

$$\begin{aligned} CDS(t,T) &= \mathbb{1}_{t<\tau}\delta\mathbb{E}\left(1 - e^{-\int_t^T \lambda_u du}\middle| \mathcal{F}_t\right) - \mathbb{1}_{t<\tau}\int_t^T \mathbb{E}\left(e^{-\int_t^s \lambda_u du}\middle| \mathcal{F}_t\right) ds \\ &= \mathbb{1}_{t<\tau}\delta\left(1 - D(t,T)\right) - \mathbb{1}_{t<\tau}\kappa\int_t^T D(t,s)ds, \end{aligned}$$

and the spread that makes the contract fair at each date t writes $\kappa\,(t,T) = \delta\,(1 - D(t,T))\,/\int_t^T D(t,s)ds$. In the example of the CIR introduced in the last section, we have

$$CDS(t,T) = \mathbb{1}_{t<\tau}\left(\delta - \delta\varphi(t,T,\lambda_t) - \kappa\int_t^T \varphi(t,s,\lambda_s)ds\right)$$

(cf. last section). For an interesting survey on the application of the CIR model to CDS pricing, the reader may refer to [7].

Example without immersion. As usual, if $(\mathcal{H})$ hypothesis does not hold, computations are more difficult and there is no simple general formulation as in the previous case, the pricing depending strongly on the form of the filtration. With the notations of the third section, in the Guo et al. context, we compute if $\widetilde{G} = \widetilde{M} - \widetilde{A}$:

$$CDS(t,T) = \mathbb{1}_{t<\tau}\frac{\mathbb{1}_{\{\widetilde{G}_t>0\}}}{\widetilde{G}_t}\delta\mathbb{E}\left(\int_t^T d\widetilde{A}_s\middle|\widetilde{\mathcal{F}}_t\right) - \mathbb{1}_{t<\tau}\frac{\mathbb{1}_{\{\widetilde{G}_t>0\}}}{\widetilde{G}_t}\kappa\mathbb{E}\left(\int_t^T \widetilde{G}_s ds\middle|\widetilde{\mathcal{F}}_t\right).$$

From $\widetilde{G}_t = D_t\Phi(Z_t,\delta,b)$[13], with the shortcut $\Phi_s = \Phi(Z_s,\delta,b)$:

$$CDS(t,T) = \frac{D_{t+\delta}}{\Phi(Z_t,\delta,b)}\mathbb{E}\left(\int_t^T \delta D_s\partial_1\Phi_s\mu\left(Z_s\right)ds + \frac{\delta D_s}{2}\partial_{1,1}\Phi_s d\left\langle Z\right\rangle_s - \kappa D_s\Phi_s ds\middle|\widetilde{\mathcal{F}}_t\right).$$

For example, if X is a Brownian Motion $x+W$, we can write:

$$\begin{aligned} CDS(t,T) &= \frac{\mathbb{1}_{t<\tau}}{\Phi(Z_t,\delta,b)}\mathbb{E}\left(\int_t^T D_s\left(\frac{\delta}{2}\partial_{1,1}\Phi_s ds - \kappa\Phi_s\right)ds\middle|\widetilde{\mathcal{F}}_t\right) \\ &= \frac{\mathbb{1}_{t<\tau}}{\Phi(Z_t,\delta,b)}\mathbb{E}\left(\int_t^T \mathbb{1}_{\inf_{t<u\le s} Z_u - Z_t > b - Z_t}\left(\frac{\delta}{2}\partial_{1,1}\Phi_s - \kappa\Phi_s\right)ds\middle|\widetilde{\mathcal{F}}_t\right) \end{aligned}$$

and finally the price writes :

$$CDS(t,T) = \frac{\mathbb{1}_{t<\tau}}{\Phi(Z_t,\delta,b)}\left(\delta a(Z_t) - \kappa b(Z_t)\right)$$

with $\Phi(x,u,b) = \mathcal{N}\left((x-b)/\sqrt{u}\right) - \mathcal{N}\left((b-x)/\sqrt{u}\right)$ and (denoting $\Phi_s^x = \Phi(W_{s-t}+x,\delta,b)$)

$$a(x) = \frac{1}{2}\mathbb{E}\left(\int_t^T \mathbb{1}_{\inf_{t<u\le s} W_{u-t} > b-x}\partial_{1,1}\Phi_s^x ds\right) \text{ and } b(x) = \mathbb{E}\left(\int_t^T \mathbb{1}_{\inf_{t<u\le s} W_{u-t} > b-x}\Phi_s^x ds\right).$$

[13] See the third part for the decomposition of this semi-martingale.

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