Basket default swaps pricing using Stein Method

Eric Benhamou   Dorinel Bastide   Marian Ciucă

Pricing Partners
www.pricingpartners.com

International Financial Research Forum
Structured Credit Products
April 28, 2008
1. CDO product
   - Definition and payoff

2. Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-$t$ Copula
   - Student Copula

3. Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method

4. Numerical Results

5. More on Stein Method
   - A brief summary
Outline

1 CDO product
   - Definition and payoff

2 Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-\(t\) Copula
   - Student Copula

3 Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method

4 Numerical Results

5 More on Stein Method
   - A brief summary
A collateralized debt obligation (CDO) is a structure of fixed income securities whose cash flows are linked to the incidence of default in a portfolio of debt instruments.
Underlying portfolio characteristics

- Number of names: $n$
- Recovery rate: $R_i$ for the $i$-th name, $i = 1, \ldots, n$
- Notional: $N_i$ for the $i$-th name, $i = 1, \ldots, n$
- Portfolio notional $N = \sum_{i=1}^{n} N_i$
Default time and Cumulative Loss

Let’s denote by $\tau_i$ the default time of the $i$-th name.

The cumulative loss at time $t$ is defined as the total loss on the portfolio:

$$L(t) = \sum_{i=1}^{n} (1 - R_i) N_i 1_{\{\tau_i \leq t\}}$$
Payoff description

Denote by $A$ and $B$ two positive numbers such that: $0 \leq A < B \leq N$.

Define by $M(t)$ the cumulative loss "tranched" between $A$ and $B$:

$$
M(t) = (L(t) - A)_+ - (L(t) - B)_+
= (L(t) - A)\mathbf{1}_{[A;B]}(L(t)) + (B - A)\mathbf{1}_{[B;N]}(L(t))
$$
CDO product

Definition and payoff

Default payments

The discounted payoff of the default payments is

\[
\text{DefLeg Payoff} = \sum_{j=1}^{n} B(\tau_j) [M(\tau_j) - M(\tau_j^{-})] \mathbf{1}_{\tau_j \leq T} = \int_{0}^{T} B(t) dM(t)
\]

Then, the price of the default leg is:

\[
\mathbb{E}\left[\int_{0}^{T} B(t) dM(t)\right] = B(T)\mathbb{E}[M(T)] + \int_{0}^{T} fw(t)B(t)\mathbb{E}[M(t)] dt,
\]

and

\[
\mathbb{E}[M(t)] = (B - A)Q_{L(t)}\{L(t) > B\} + \int_{0}^{T} (x - A)Q_{L(t)}(dx),
\]

where \(Q_{L(t)}\) is the the distribution of \(L(t)\).
The Premium Leg is the sum of the Regular Payments and of the Accrued Coupon

The Regular Payments Payoff:

\[
\sum_{i=1}^{l} B(t_i) (t_i - t_{i-1}) X [M(\infty) - M(t_i)]
\]

The Accrued Coupon:

\[
\sum_{j=1}^{n} B(\tau_j) (\tau_j - t_{k(j)-1}) [M(\tau_j) - M(\tau_j^-)] 1_{t_{k(j)-1} < \tau_j < t_{k(j)}}.
\]
Outline

1. CDO product
   - Definition and payoff

2. Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-$t$ Copula
   - Student Copula

3. Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method

4. Numerical Results

5. More on Stein Method
   - A brief summary
To generate a one-factor model for the $\tau_i \ (1 \leq i \leq N),$ we define the following random variables $X_i \ (1 \leq i \leq N)$:

$$X_i = \rho_i V + \sqrt{1 - \rho_i^2} \bar{V}_i,$$

where $V$ and the $\bar{V}_i$ are independent standard Gaussian random variables and $-1 \leq \rho_i < 1.$ This Equation defines a correlation structure between the $X_i$’s dependent on a single common factor $V.$
Under the copula model, each $X_i$ is mapped to $\tau_i$ using a percentile-to-percentile transformation:

$$Q(\tau_i \leq t \mid V) = Q(X_i \leq x \mid V)$$

$$= \Phi \left( -\rho_i V + \Phi^{-1}(F_i(t)) \frac{1}{\sqrt{1 - \rho_i^2}} \right)$$
One Factor Clayton Copula Model

Let $V$ a random variable following a standard Gamma distribution with shape parameter $1/\theta$ ($\theta > 0$) and scale parameter equal to 1. Its probability density is:

$$f(x) = x^{(1-\theta)/\theta} \frac{e^{-x}}{\Gamma(1/\theta)}.$$
One Factor Clayton Copula Model

We then define:

$$X_i = \left(1 - \frac{\ln(U_i)}{V}\right)^{-1/\theta},$$

where $U_1, \cdots, U_N$ are independent uniform random variables also independent from $V$.

The conditional default probabilities are:

$$p_t^{i|V} = \exp \left\{ V \left(1 - F_i(t)^{-\theta}\right) \right\}.$$
One Factor Double-$t$ Copula Model

Let $\nu$ be an integer. The Student’s $t$-distribution with degree of freedom $\nu$ is given by

$$s_\nu(x) = \frac{\Gamma((\nu + 1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}.$$

For integer numbers $\nu$ and $\bar{\nu}$, define

$$X_i = \rho \left(\frac{\nu - 2}{\nu}\right)^{1/2} V + \sqrt{1 - \rho^2} \left(\frac{\bar{\nu} - 2}{\bar{\nu}}\right)^{1/2} \bar{V}_i,$$

where $V$ is a Student random variable with degree of freedom $\nu$, and $\bar{V}_i (1 \leq i \leq n)$ are Student random variables with degree of freedom $\bar{\nu}$. 
Denote by $t_\nu$ the cumulative distribution function of a Student random variable with $\nu$ degree of freedom. The conditional default probabilities are given by:

$$p_t^{i|\nu} = t_\nu \left( \left( \frac{\nu}{\nu - 2} \right)^{1/2} H_i^{-1}(F_i(t)) - \rho \left( \frac{\nu - 2}{\nu} \right)^{1/2} \sqrt{1 - \rho^2} \frac{V}{V} \right),$$

where $H_i$ is the c.d.f. of $X_i$, and $H_i^{-1}$ is its inverse.
2 Factors Student Copula Model

Let $V_1$ be a normal random variable, and, for an integer $\nu$, let $V_2$ be an inverse gamma random variable with scale and shape parameters both equal to $\nu/2$ independent from $V_1$.

We define

$$X_i = \sqrt{V_2} \left( \rho V_1 + \sqrt{1 - \rho^2} \ V_i \right),$$

where $(V_i)_{1 \leq i \leq n}$ are independent Normal random variables also independent from $V_1$ and $V_2$. 
Then, the conditional default probability given the factors \((V_1, V_2)\) are

\[
\rho_t^{i|V_1, V_2} = \Phi \left( \frac{-\rho V_1 + \frac{1}{\sqrt{V_2}} t_{\nu}^{-1}(F_i(t))}{\sqrt{1 - \rho^2}} \right)
\]
Outline

1. CDO product
   - Definition and payoff

2. Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-$t$ Copula
   - Student Copula

3. Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method

4. Numerical Results

5. More on Stein Method
   - A brief summary

Benhamou, Bastide, Ciucă (PP)
Let us compute, at point $u$, the probability generating function of $L(t)$:

$$\psi_{L(t)}(u) = \mathbb{E} \left[ \mathbb{E} \left\{ u^{L(t)} \mid V \right\} \right]$$

$$= \mathbb{E} \left[ \mathbb{E} \left\{ u \sum_{i=1}^{n} (1-R_i)N_i 1\{\tau_i \leq t\} \mid V \right\} \right]$$

$$= \mathbb{E} \left[ \prod_{j=1}^{n} \left( q^{|V|}_j t + p^{|V|}_j t u^{M_j} \right) \right].$$
Probability Generating Function

Let $l_1, \ldots, l_m$ be the set of values of $L(t)$. The PGF method computes, for $i = 1, \ldots, m$:

- the value $l_i$
- the coefficient $C_{l_i}$ of $u^l_i$

Then $C_{l_i} = \mathbb{Q}(L(t) = l_i)$, for $i = 1, \ldots, m$.

The PGF method is exact: the whole discrete distribution of $L(t)$ is exactly computed.

Observe that the Recursive method of Hull and White approximates the distribution of $L(t)$. 
A first intuitive way is to approximate the cumulative loss distribution by its moment matching normal distribution. The first moment of the corresponding normal, denoted by $\mu_{L(t)|V}$, is:

$$\mu_{L(t)|V} = \mathbb{E} (L(t)|V) = \sum_{i=1}^{n} (1 - R_i) N_i p_t^{i|V},$$

where $p_t^{i|V} = \mathbb{Q} (\tau_i \leq t|V)$ for $i = 1, \ldots, n$. The variance is also provided by moment matching:

$$\sigma^2_{L(t)|V} = \text{Var} (L(t)|V) = \sum_{i=1}^{n} (1 - R_i)^2 N_i^2 p_t^{i|V} - \mu^2_{L(t)|V}. $$
The call-spread on the cumulative loss can be computed in closed form:

$$
\mathbb{E} \left( (L(t) - A)_+ \mid V \right) = \sigma_{L(t)\mid V} \varphi \left( \frac{\mu_{L(t)\mid V} - A}{\sigma_{L(t)\mid V}} \right) \\
+ \left( \mu_{L(t)\mid V} - A \right) \Phi \left( \frac{\mu_{L(t)\mid V} - A}{\sigma_{L(t)\mid V}} \right),
$$

where $\varphi$ and $\Phi$ are the density and the cumulative distribution function of the standard Gaussian distribution.
The accuracy of the previous method is not very good: only the first two moments are matched, which leads to a crude approximation.

Recently El Karoui, Jiao and Kurtz (2007) introduced a new numerical method, based on Stein’s method and zero bias transformation, to compute

\[ \mathbb{E}\{(L(t) - A)_+\}. \]

They introduced first order correction terms for both Gaussian and Poisson approximations, and discussed the error approximation.
The intuition of this method is to find, for the cumulative loss distribution $L(t)$, an expansion around a well known density with closed form for the call spread.

With $h(x) = (x - A)_+$, suppose that we approximate the law of $L(t)$ with the law of another random variable $Z$, and want to know how far $\mathbb{E}h(Z)$ is from $\mathbb{E}h(L(t))$. In their paper El Karoui, Jiao and Kurtz proved that:

$$\mathbb{E}h(L(t)) = \mathbb{E}h(Z) + C(h, A, t) + \varepsilon(h, A, t),$$

where $C(h, A, t)$ is a corrector term, and the corrected approximation error $\varepsilon(h, A, t)$ is bounded.
Stein Method: Normal law

Let $X_i, i = 1, \ldots, n$ be $n$ Bernoulli distributed r. v., with parameter $p$.

The Central Limit Theorem tells us that the asymptotic law for the sum

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - p)$$

is the normal distribution $\mathcal{N}(0, p(1 - p))$. Then, for large $n$,

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(np, np(1 - p))$$

where the Bernoulli parameter is given by $p = \mathbb{E}(X_i)$, for $i = 1, \ldots, n$. 
Stein Method: Poisson law

Notice also that the characteristic function of the sum is \( u \in [0; 1] \):

\[
\mathbb{E} \left( e^{iu \sum_{j=1}^{n} X_j} \right) = \left( pe^{iu} + (1 - p) \right)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{p^k (e^{iu} - 1)^k}{k!}.
\]

Observe that a Poisson random variable \( Y \) with intensity parameter equal to \( np \) has its characteristic function given by:

\[
\mathbb{E} \left( e^{iuY} \right) = e^{np(e^{iu} - 1)} = \sum_{k=0}^{\infty} n^k p^k \frac{(e^{iu} - 1)^k}{k!} \approx \sum_{k=0}^{n} n^k p^k \frac{(e^{iu} - 1)^k}{k!}
\]

Hence, considering that \( \frac{n!}{(n-k)!} \) is not far from \( n^k \), we see intuitively that the sum distribution can be approximated by a Poisson law:

\[
\sum_{i=1}^{n} X_i \sim \mathcal{P}(np).
\]
The tranche loss can be approximated with the following normal formula

\[ \mathbb{E}((L(t) - A)_+ | V) \approx \mathbb{E}((Z(t) - \tilde{A})_+ | V) + C_{(.-A)_+}^N(t) \]

where

\[ Z(t) \sim \mathcal{N}(\mu_{L(t)|V}, \sigma_{L(t)|V}^2). \]

The normal corrector \( C_{(.-A)_+}^N \) is given by

\[ C_{(.-A)_+}^N(t) = \frac{\tilde{A} \phi_{\sigma_{L(t)|V}}(\tilde{A})}{6\sigma_{L(t)|V}^2} Q(\tau_i \leq t | V). \]
The Poisson corrector $\mathcal{C}_P^{(\cdot - \tilde{A})_+}$ is given by:

$$
\mathcal{C}_P^{(\cdot - \tilde{A})_+}(t) = \frac{\sigma^2}{2} \frac{L(t) V - \lambda V(t)}{\left( \frac{A}{(1-R)N} - 1 \right)} e^{-\lambda V(t)} \lambda V(t) \left[ \frac{A}{(1-R)N} \right]^{-1},
$$

and

$$
\mathbb{E}((Z(t) - \tilde{A})_+ | V) = \lambda V(t) - \tilde{A} - e^{-\lambda V(t)} \sum_{k=1}^{\lceil \tilde{A} \rceil - 1} (k - \lceil \tilde{A} \rceil) \frac{(\lambda V(t))^k}{k!}.
$$
What is the best solution?

Choose the best approximation following the condition: if

\[ \mu_L(t) | V < 15 \]

then apply Poisson approximation, otherwise apply Normal approximation.

See El Karoui, Jiao and Kurtz for full details.
Outline

1. CDO product
   - Definition and payoff

2. Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-$t$ Copula
   - Student Copula

3. Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method

4. Numerical Results

5. More on Stein Method
   - A brief summary
Numerical Results

Gaussian copula

Stein vs Recursive with Gaussian Copula, 125 size pool

Break Even Error in Basis Points

-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0

[0; 3] [3; 7] [7; 10] [10; 15] [15; 30] [30; 60] [0; 100]

Stein vs Recursive with Gaussian Copula, 125 size pool

-3.0 -2.0 -1.0 0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0

[0; 3] [3; 7] [7; 10] [10; 15] [15; 30] [30; 60] [0; 100]

Benhamou, Bastide, Ciucă (PP)
Numerical Results

Clayton copula

Stein vs PGF with Clayton Copula, 125 size pool, 5Y time horizon

Break Even Error in Basis Points

theta=0.01
theta=0.05
theta=0.18
theta=0.36
theta=0.66
Numerical Results

Student copula with 12 degrees of freedom

Stein vs Recursive with Student Copula 12 degrees of freedom, 125 size pool

Break Even Error in Basis Points

rho=47%  rho=67%  rho=82%  rho=100%

Benhamou, Bastide, Ciucă (PP)  Stein Method  Structured Credit Products  34 / 47
Double-t copula with 12 degrees of freedom

Stein vs Recursive with Double-t Copula, rho=58% and 12 degrees of freedom, 125 size pool

Break Even Error in Basis Points

Stein Method

Benhamou, Bastide, Ciucă (PP)
Stein versus Normal Approximation

Numerical Results

Stein vs Normal with Gaussian Copula \(\rho=1\%\), 125 size pool, 5Y time horizon

Stein vs Normal with Gaussian Copula \(\rho=32\%\), 125 size pool, 5Y time horizon

Stein vs Normal with Gaussian Copula \(\rho=55\%\), 125 size pool, 5Y time horizon

Stein vs Normal with Gaussian Copula \(\rho=84\%\), 125 size pool, 5Y time horizon

Break Even Error in Basis Points

Stein Normal Approx
### Numerical Results

#### Computation time

<table>
<thead>
<tr>
<th>Method/Copula</th>
<th>Gaussian</th>
<th>Clayton</th>
<th>Student</th>
<th>Double-t</th>
</tr>
</thead>
<tbody>
<tr>
<td>PGF max</td>
<td>30.67 sec</td>
<td>30.05 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Recursive max</td>
<td>3.41 sec</td>
<td>2.93 sec</td>
<td>183.43 sec</td>
<td>335 sec</td>
</tr>
<tr>
<td>Stein max</td>
<td>0.68 sec</td>
<td>0.57 sec</td>
<td>119.18 sec</td>
<td>332 sec</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Best Ratio Recursive/Stein</td>
<td></td>
<td></td>
<td>Gaussian</td>
<td>Clayton</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>15.65</td>
<td>20</td>
</tr>
</tbody>
</table>
Outline

1. CDO product
   - Definition and payoff

2. Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-$t$ Copula
   - Student Copula

3. Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method

4. Numerical Results

5. More on Stein Method
   - A brief summary
Stein’s method and zero bias transformation: Application to CDOs pricing

Nicole El Karoui\textsuperscript{1} Ying Jiao\textsuperscript{1,2}

\textsuperscript{1}CMAP Ecole Polytechnique

\textsuperscript{2}Ecole supérieure d’ingénieurs Léonard de Vinci

International Financial Research Forum
Structured Credit Products
April 28, 2008
A brief summary of theoretical results

1. **Practical motivation**: provide a rapid and robust numerical method in the market-adopted framework, for pricing and risk management

2. **Difficulty** for the existing approximation methods: the normal approximation and the saddle-point method

3. **Solution**: combine the normal and the Poisson approximations and propose higher-order formulas in both cases

4. **Mathematical methods**: Stein’s method and zero bias transformation
Stein’s method and zero bias transformation

- **Advantage** of Stein’s method is its flexibility for both normal and Poisson distributions.

- Observation of Stein and Chen: a reference law $\mu$ can be characterized by the associated operator $A_\mu$, i.e. for a r.v. $Z \sim \mu$,

$$
\mathbb{E}[Zf(Z)] - \mathbb{E}[A_\mu f(Z)] = 0.
$$

(1)

- In the central normal case, $A_N f(x) = \sigma^2 f'(x)$. In the Poisson case, $A_P f(x) = \lambda f(x + 1)$.

- Zero bias transformation is introduced in this context: for a r.v. $X$, $X^*$ is said to have the $X$-zero biased distribution of $X$ if

$$
\mathbb{E}[Xf(X)] = \mathbb{E}[A_\mu f(X^*)] = 0.
$$

(2)
Stein’s method and zero bias transformation

- Stein’s equation concerns, for a given function \( h \), an auxiliary function \( f \):
  \[
  h(x) - \int h \, d\mu = xf(x) - A_\mu f(x). 
  \]  

- Combine (2) and (3), we have
  \[
  \mathbb{E}[h(X)] - \int h \, d\mu = \mathbb{E}[A_\mu f_h(X^*) - A_\mu f_h(X)] 
  \]  

  où \( f_h \) est la solution de (3).

- To obtain efficient error estimation for the approximations, it’s crucial to estimate the difference between \( X \) and \( X^* \) and some superior norm of \( A_\mu f_h \) and its derivatives.
First order correctors

- We obtain a first-order corrector for the direct normal or Poisson approximation in the above framework for the sum of independent random variables.

- The normal corrector for $\mathbb{E}[h(W)]$ where $W = X_1 + \cdots + X_n$ is

$$C_h = \frac{1}{\sigma^2_W} \mathbb{E}[X_i^*] \Phi_{\sigma_W} \left( \left( \frac{x^2}{3\sigma^2_W} - 1 \right) x h(x) \right)$$

where $i$ is a random index on the set $\{1, \cdots, n\}$ and $\Phi_{\sigma}$ is the normal expectation function.

- The Poisson corrector is of similar form:

$$C_h^P = \frac{\lambda V}{2} \mathbb{E}[Y_i^* - Y_i] \mathcal{P}_{\lambda V}(\Delta^2 h)$$

where $\Delta$ is the difference function. Here $Y_i$ are integer valued.
Some remarks and perspective

- The formulas are simple and hence rapid to calculate numerically, especially for the call function \( h(x) = (x - k)^+ \)
- The methodology applies to all factor models where the conditional independence assumption holds.
- From the theoretical point of view, it’s important to estimate the approximation errors, where the call function is more difficult to treat. The corrected approximation is of same order as in the symmetric case and is of \( O(1/n) \) in the i.i.d. case.
- **Perspective**: generalize the method to the continuous Poisson case and treat the stochastic recoveries in the Poisson approximation.
References

- J.-P. Laurent, J. Gregory (2003), *Basket Default Swaps, CDO’s and Factor Copulas*. 

El Karoui, Jiao (CMAP and ESILV)
Contact

Marian Ciucă,  Marian.Ciuca@PricingPartners.com

Ying Jiao,  jiaoying@cmapx.polytechnique.fr
Thank you for your attention!