Basket default swaps pricing using Stein Method

Eric Benhamou Dorinel Bastide Marian Ciucă

Pricing Partners
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Outline

1. CDO product
   - Definition and payoff
2. Factor Copula Models
   - Gaussian Copula
   - Clayton Copula
   - Double-*t* Copula
   - Student Copula
3. Numerical Methods
   - Exact FFT-like method
   - Normal Approximation
   - Stein Method
4. Numerical Results
5. More on Stein Method
   - A brief summary

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CDO product

A collateralized debt obligation (CDO) is a structure of fixed income securities whose cash flows are linked to the incidence of default in a portfolio of debt instruments.

Underlying portfolio characteristics

- Number of names: n
- Recovery rate: R_i for the i-th name, $i = 1, ..., n$
- Notional: N_i for the i-th name, $i = 1, ..., n$
- Portfolio notional $N = \sum_1^n N_i$

Default time and Cumulative Loss

Let's denote by τ_i the default time of the i-th name.

The cumulative loss at time t is defined as the total loss on the portfolio:

$$L(t) = \sum_{i=1}^{n} (1 - R_i) \, N_i \mathbf{1}_{\{\tau_i \leq t\}}$$

Payoff description

Denote by A and B two positive numbers such that: $0 \leq A < B \leq N$.

Define by $M(t)$ the cumulative loss "tranched" between A and B:

$$\begin{aligned} M(t) &= (L(t) - A)_+ - (L(t) - B)_+ \\ &= (L(t) - A)\mathbf{1}_{[A;B]}(L(t)) + (B - A)\mathbf{1}_{]B;N]}(L(t)) \end{aligned}$$

Default payments

The discounted payoff of the default payments is

$$\begin{aligned}\text{DefLeg Payoff} &= \sum_{j=1}^{n} B(\tau_j)\left[M(\tau_j) - M(\tau_j^-)\right]\mathbf{1}_{\tau_j \leq T} = \int_0^T B(t)dM(t) \\ &= B(t)M(t) + \int_0^T fw(t)B(t)M(t)dt.\end{aligned}$$

Then, the price of the default leg is:

$$\mathbb{E}\left[\int_0^T B(t)dM(t)\right] = B(T)\mathbb{E}[M(T)] + \int_0^T fw(t)B(t)\mathbb{E}[M(t)]dt,$$

and

$$\mathbb{E}[M(t)] = (B-A)Q_{L(t)}\{L(t) > B\} + \int_0^T (x-A)Q_{L(t)}(dx),$$

where $Q_{L(t)}$ is the the distribution of $L(t)$.

Premium payments

- The Premium Leg is the sum of the Regular Payments and of the Accrued Coupon
- The Regular Payments Payoff:

$$\sum_{i=1}^{I} B(t_i)\,(t_i - t_{i-1})\,X\,[M(\infty) - M(t_i)]$$

- The Accrued Coupon:

$$\sum_{j=1}^{n} B(\tau_j)\,(\tau_j - t_{k(j)-1})\,[M(\tau_j) - M(\tau_j^-)]\,\mathbf{1}_{t_{k(j)-1}<\tau_j<t_{k(j)}}.$$

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One Factor Gaussian Copula Model

To generate a one-factor model for the τ_i $(1 \le i \le N)$, we define the following random variables X_i $(1 \le i \le N)$:

$$X_i = \rho_i V + \sqrt{1 - \rho_i^2}\, \overline{V}_i,$$

where V and the $\overline{V}_i$ are independent standard Gaussian random variables and $-1 \le \rho_i < 1$. This Equation defines a correlation structure between the X_i's dependent on a single common factor V.

One Factor Gaussian Copula Model

Under the copula model, each X_i is mapped to τ_i using a percentile-to-percentile transformation:

$$\begin{aligned}\mathbb{Q}(\tau_i \leq t|V) &= \mathbb{Q}(X_i \leq x \mid V) \\ &= \Phi\left(\frac{-\rho_i V + \Phi^{-1}(F_i(t))}{\sqrt{1-\rho_i^2}}\right)\end{aligned}$$

One Factor Clayton Copula Model

Let V a random variable following a standard Gamma distribution with shape parameter $1/\theta$ ($\theta > 0$) and scale parameter equal to 1. Its probability density is:

$$f(x) = x^{(1-\theta)/\theta} \frac{e^{-x}}{\Gamma(1/\theta)}.$$

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One Factor Clayton Copula Model

We then define:

$$X_i = \left(1 - \frac{\ln(U_i)}{V}\right)^{-1/\theta},$$

where $U_1, \cdots, U_N$ are independent uniform random variables also independent from V.

The conditional default probabilities are:

$$p_t^{i|V} = \exp\left\{V\left(1 - F_i(t)^{-\theta}\right)\right\}.$$

One Factor Double-t Copula Model

Let ν be an integer. The Student's t-distribution with degree of freedom ν is given by

$$s_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2}.$$

For integer numbers ν and $\overline{\nu}$, define

$$X_i = \rho\left(\frac{\nu-2}{\nu}\right)^{1/2} V + \sqrt{1-\rho^2}\left(\frac{\overline{\nu}-2}{\overline{\nu}}\right)^{1/2}\overline{V}_i,$$

where V is a Student random variable with degree of freedom ν, and $\overline{V}_i$ $(1 \leq i \leq n)$ are Student random variables with degree of freedom $\overline{\nu}$.

One Factor Double-t Copula Model

Denote by t_ν the cumulative distribution function of a Student random variable with ν degree of freedom. The conditional default probabilities are given by:

$$p_t^{i|V} = t_{\overline{\nu}}\left(\left(\frac{\overline{\nu}}{\overline{\nu}-2}\right)^{1/2}\frac{H_i^{-1}(F_i(t)) - \rho\left(\frac{\nu-2}{\nu}\right)^{1/2}V}{\sqrt{1-\rho^2}}\right),$$

where H_i is the c.d.f. of X_i, and H_i^{-1} is its inverse.

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2 Factors Student Copula Model

Let V_1 be a normal random variable, and, for an integer ν, let V_2 be an inverse gamma random variable with scale and shape parameters both equal to $\nu/2$ independent from V_1.

We define

$$X_i = \sqrt{V_2}\left(\rho V_1 + \sqrt{1-\rho^2}\,\overline{V}_i\right),$$

where $(\overline{V}_i)_{1\leq i\leq n}$ are independent Normal random variable also independent from V_1 and V_2.

2 Factors Student Copula Model

Then, the conditional default probability given the factors $(V_1,\ V_2)$ are

$$p_t^{i|V_1,V_2} = \Phi\left(\frac{-\rho V_1 + \frac{1}{\sqrt{V_2}}\ t_\nu^{-1}(F_i(t))}{\sqrt{1-\rho^2}} \right)$$

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Probability Generating Function (PGF)

Let us compute, at point u, the probability generating function of $L(t)$:

$$\begin{aligned}\psi_{L(t)}(u) &= \mathbb{E}\left[\mathbb{E}\left\{u^{L(t)}|V\right\}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left\{u^{\sum_{i=1}^{n}(1-R_i)N_i\mathbf{1}_{\{\tau_i\leq t\}}}|V\right\}\right] \\ &= \mathbb{E}\left[\prod_{j=1}^{n}\left(q_t^{j|V}+p_t^{j|V}u^{M_j}\right)\right].\end{aligned}$$

Probability Generating Function

Let $l_1, ..., l_m$ be the set of values of $L(t)$. The PGF method computes, for $i = 1, ..., m$:

- the value l_i
- the coefficient C_{l_i} of u^{l_i}

Then $C_{l_i} = \mathbb{Q}(L(t) = l_i)$, for $i = 1, ..., m$.

The PGF method is exact: the whole discrete distribution of $L(t)$ is exactly computed.

Observe that the Recursive method of Hull and White *approximates* the distribution of $L(t)$.

Normal Approximation Method

A first intuitive way is to approximate the cumulative loss distribution by its moment matching normal distribution. The first moment of the corresponding normal, denoted by $\mu_{L(t)|V}$, is:

$$\mu_{L(t)|V} = \mathbb{E}\left(L(t)|V\right) = \sum_{i=1}^{n} \left(1 - R_i\right) N_i p_t^{i|V},$$

where $p_t^{i|V} = \mathbb{Q}\left(\tau_i \leq t|V\right)$ for $i = 1, \ldots, n$. The variance is also provided by moment matching:

$$\sigma^2_{L(t)|V} = \text{Var}\left(L(t)|V\right) = \sum_{i=1}^{n} \left(1 - R_i\right)^2 N_i^2 p_t^{i|V} - \mu^2_{L(t)|V}.$$

Normal Approximation Method

The call-spread on the cumulative loss can be computed in closed form:

$$\begin{aligned}\mathbb{E}\left(\left(L(t)-A\right)_+ \,|\, V\right) &= \sigma_{L(t)|V}\varphi\left(\frac{\mu_{L(t)|V}-A}{\sigma_{L(t)|V}}\right) \\ &\quad + \left(\mu_{L(t)|V}-A\right)\Phi\left(\frac{\mu_{L(t)|V}-A}{\sigma_{L(t)|V}}\right),\end{aligned}$$

where φ and Φ are the density and the cumulative distribution function of the standard Gaussian distribution.

Stein Method

The accuracy of the previous method is not very good: only the first two moments are matched, which leads to a crude approximation.

Recently El Karoui, Jiao and Kurtz (2007) introduced a new numerical method, based on Stein's method and zero bias transformation, to compute

$$\mathbb{E}\{(L(t) - A)_+\}.$$

They introduced first order correction terms for both Gaussian and Poisson approximations, and discussed the error approximation.

Stein Method

The intuition of this method is to find, for the cumulative loss distribution $L(t)$, an expansion around a well known density with closed form for the call spread.

With $h(x) = (x - A)_+$, suppose that we approximate the law of $L(t)$ with the law of another random variable Z, and want to know how far $\mathbb{E}h(Z)$ is from $\mathbb{E}h(L(t))$. In their paper El Karoui, Jiao and Kurtz proved that:

$$\mathbb{E}h(L(t)) = \mathbb{E}h(Z) + C(h, A, t) + \varepsilon(h, A, t),$$

where $C(h, A, t)$ is a corrector term, and the corrected approximation error $\varepsilon(h, A, t)$ is bounded.

Stein Method: Normal law

Let $X_i,\ i = 1, ..., n$ be n Bernoulli distributed r. v., with parameter p.

The Central Limit Theorem tells us that the asymptotic law for the sum $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - p)$ is the normal distribution $\mathcal{N}(0, p(1-p))$. Then, for large n,

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(np, np(1-p))$$

where the Bernoulli parameter is given by $p = \mathbb{E}(X_i)$, for $i = 1, \ldots, n$.

Stein Method: Poisson law

Notice also that the characteristic function of the sum is ($u \in]0;1[$):

$$\mathbb{E}\left(e^{iu\sum_{j=1}^{n} X_j}\right) = \left(pe^{iu} + (1-p)\right)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{p^k \left(e^{iu}-1\right)^k}{k!}.$$

Observe that a Poisson random variable Y with intensity parameter equal to np has its characteristic function given by:

$$\mathbb{E}\left(e^{iuY}\right) = e^{np\left(e^{iu}-1\right)} = \sum_{k=0}^{\infty} n^k \frac{p^k \left(e^{iu}-1\right)^k}{k!} \simeq \sum_{k=0}^{n} n^k \frac{p^k \left(e^{iu}-1\right)^k}{k!}$$

Hence, considering that $\frac{n!}{(n-k)!}$ is not far from n^k, we see intuitively that the sum distribution can be approximated by a Poisson law:

$$\sum_{i=1}^{n} X_i \sim \mathcal{P}(np).$$

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Stein Method: Normal law

The tranche loss can be approximated with the following normal formula

$$\mathbb{E}((L(t) - A)_+ | V) \approx \mathbb{E}((Z(t) - \widetilde{A})_+ | V) + \mathcal{C}^{\mathcal{N}}_{(.-A)_+}(t)$$

where

$$Z(t) \sim \mathcal{N}(\mu_{L(t)|V}, \sigma^2_{L(t)|V}).$$

The normal corrector $\mathcal{C}^{\mathcal{N}}_{(.-A)_+}$ is given by

$$\mathcal{C}^{\mathcal{N}}_{(.-A)_+}(t) = \frac{\widetilde{A}\phi_{\sigma_{L(t)|V}}(\widetilde{A})}{6\sigma^2_{L(t)|V}} \mathbb{Q}(\tau_i \leq t | V).$$

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Stein Method: Poisson law

The Poisson corrector $\mathcal{C}^{\mathcal{P}}_{(.-\widetilde{A})_+}$ is given by:

$$\mathcal{C}^{\mathcal{P}}_{(.-\widetilde{A})_+}(t) = \frac{\sigma^2_{\widetilde{L}(t)|V} - \lambda_V(t)}{2\left(\left\lfloor \frac{A}{(1-R)\overline{N}} \right\rfloor - 1\right)!} e^{-\lambda_V(t)} \lambda_V(t)^{\left\lfloor \frac{A}{(1-R)\overline{N}} \right\rfloor - 1},$$

and

$$\mathbb{E}((Z(t) - \widetilde{A})_+ | V) = \lambda_V(t) - \widetilde{A} - e^{-\lambda_V(t)} \sum_{k=1}^{\lfloor \widetilde{A} \rfloor - 1} \left(k - \lfloor \widetilde{A} \rfloor\right) \frac{(\lambda_V(t))^k}{k!}.$$

What is the best solution?

Choose the best approximation following the condition: if

$$\mu_{L(t)|V} < 15$$

then apply Poisson approximation, otherwise apply Normal approximation.

See El Karoui, Jiao and Kurtz for full details.

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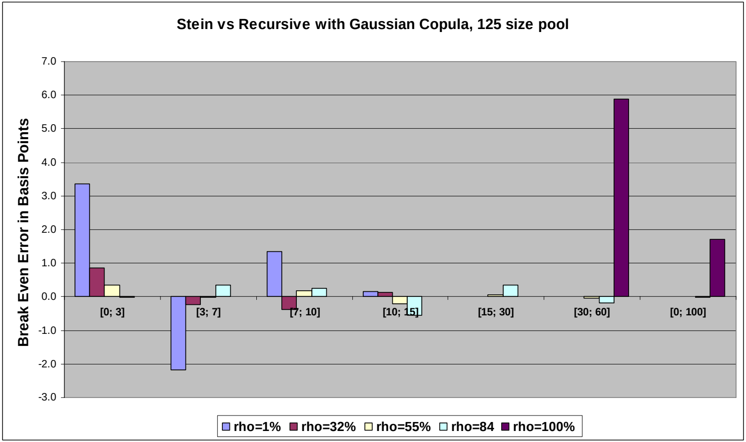
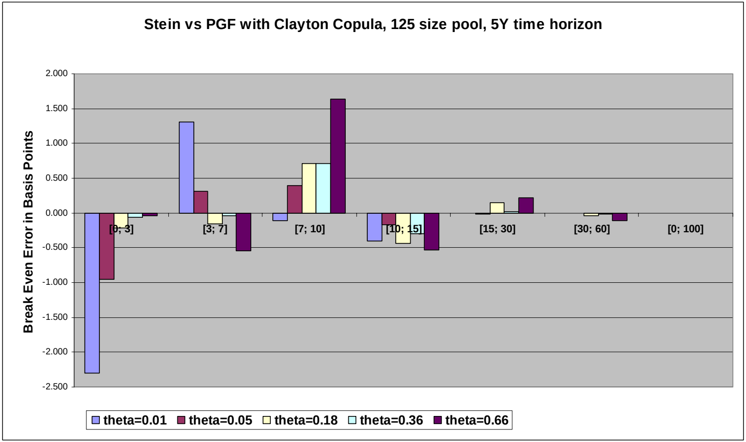
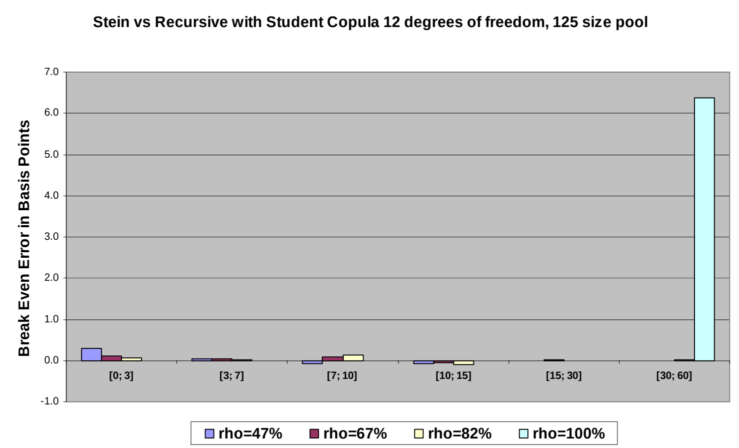
Gaussian copula



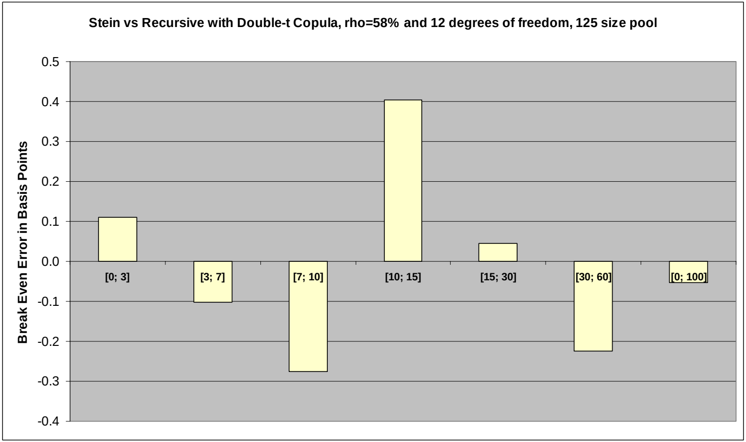
Clayton copula



Student copula with 12 degrees of freedom



Double-t copula with 12 degrees of freedom



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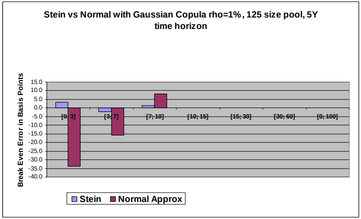
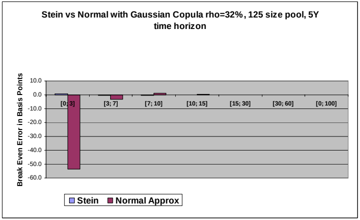
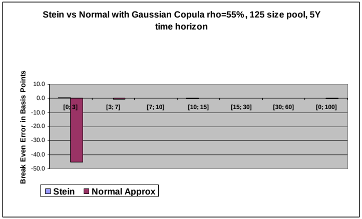
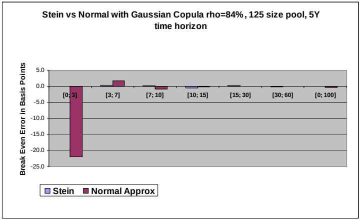
Stein versus Normal Approximation









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Computation time

Method/Copula	Gaussian	Clayton	Student	Double-t
PGF max	30.67 sec	30.05 sec		
Recursive max	3.41 sec	2.93 sec	183.43 sec	335 sec
Stein max	0.68 sec	0.57 sec	119.18 sec	332 sec

	Gaussian	Clayton
Best Ratio Recursive/Stein	15.65	20

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Stein's method and zero bias transformation: Application to CDOs pricing

Nicole El Karoui[1] Ying Jiao[1,2]

[1]CMAP Ecole Polytechnique

[2]Ecole supérieure d'ingénieurs Léonard de Vinci

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A brief summary of theoretical results

1. **Practical motivation**: provide a rapid and robust numerical method in the market-adopted framework, for pricing and risk management
2. **Difficulty** for the existing approximation methods: the normal approximation and the saddle-point method
3. **Solution**: combine the normal and the Poisson approximations and propose higher-order formulas in both cases
4. **Mathematical methods**: Stein's method and zero bias transformation

Stein's method and zero bias transformation

- **Advantage** of Stein's method is its flexibility for both normal and Poisson distributions.
- Observation of Stein and Chen: a reference law μ can be characterized by the associated operator $\mathcal{A}_\mu$, i.e. for a r.v. $Z \sim \mu$,

$$\mathbb{E}[Zf(Z)] - \mathbb{E}[\mathcal{A}_\mu f(Z)] = 0. \tag{1}$$

- In the central normal case, $\mathcal{A}_N f(x) = \sigma^2 f'(x)$. In the Poisson case, $\mathcal{A}_P f(x) = \lambda f(x+1)$.
- Zero bias transformation is introduced in this context: for a r.v. X, X^* is said to have the X-zero biased distribution of X if

$$\mathbb{E}[Xf(X)] = \mathbb{E}[\mathcal{A}_\mu f(X^*)] = 0. \tag{2}$$

Stein's method and zero bias transformation

- Stein's equation concerns, for a given function h, an auxiliary function f:
$$h(x) - \int h d\mu = xf(x) - \mathcal{A}_\mu f(x). \tag{3}$$
- Combine (2) and (3), we have
$$\mathbb{E}[h(X)] - \int h d\mu = \mathbb{E}[\mathcal{A}_\mu f_h(X^*) - \mathcal{A}_\mu f_h(X)] \tag{4}$$
où f_h est la solution de (3).
- To obtain efficient error estimation for the approximations, it's crucial to estimate the difference between X and X^* and some superior norm of $\mathcal{A}_\mu f_h$ and its derivatives.

First order correctors

- We obtain a first-order corrector for the direct normal or Poisson approximation in the above framework for the sum of independent random variables.
- The normal corrector for $\mathbb{E}[h(W)]$ where $W = X_1 + \cdots + X_n$ is

$$C_h = \frac{1}{\sigma_W^2}\mathbb{E}[X_I^*]\Phi_{\sigma_W}\Big(\big(\frac{x^2}{3\sigma_W^2} - 1\big)xh(x)\Big)$$

where I is a random index on the set $\{1, \cdots, n\}$ and Φ_σ is the normal expectation function.

- The Poisson corrector is of similar form:

$$C_h^P = \frac{\lambda_V}{2}\mathbb{E}[Y_I^* - Y_I]\mathcal{P}_{\lambda_V}(\Delta^2 h)$$

where Δ is the difference function. Here Y_i are integer valued.

Some remarks and perspective

- The formulas are simple and hence rapid to calculate numerically, especially for the call function $h(x) = (x - k)^+$
- The methodology applies to all factor models where the conditional independence assumption holds.
- From the theoretical point of view, it's important to estimate the approximation errors, where the call function is more difficult to treat. The corrected approximation is of same order as in the symmetric case and is of $O(1/n)$ in the i.i.d. case.
- **Perspective**: generalize the method to the continuous Poisson case and treat the stochastic recoveries in the Poisson approximation.

References

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Contact

Marian Ciucă, Marian.Ciuca@PricingPartners.com

Ying Jiao, jiaoying@cmapx.polytechnique.fr

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Thank you for your attention!