Model-independent approximation formulas for ABS duration

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International Financial Research Forum – March 28th, 2008

SGAM Alternative Investments

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Asset Management

Context

- **Asset Backed Securities: MBS, granular ABS, CDO**
- **Prepayment risk**
  - Turnover
  - Refinancing
- **Pricing models**
  - Option based models : valuing the optimal refinancing option
  - Econometric models : prepayment rate depends on some explicative variables such as interest rates
  - Hybrid models

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ABS trading

■ Asset picking

- Fundamental analysis (macroeconomic analysis, analysis of the credit enhancement mechanisms)
- Quantitative analysis: pricing the Option Adjusted Spread (OAS) and rating agencies models

■ Mesure du rendement d’un ABS

- Cash-flow model
- Constant prepayment rate
- Discount Margin (constant spread)
- WAL = proxy of the duration

■ Price = sum of the discounted cash-flows in the 0 default scenario

$$P = \sum_t \frac{CF_t}{(r_t + DM)^t}$$

Our results

New formalism for prepayments

- Price = sum of discounted cash-flows
- Price = sum of discount factors weighted by cash-flows

Approximation formulas

- Duration and convexity as a function of
  - WAL
  - Price
  - Spread
  - Market spread (Discount Margin)

Accuracy tests

New formalism for prepayments

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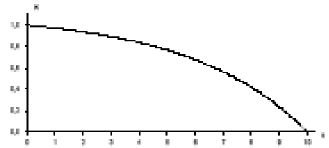
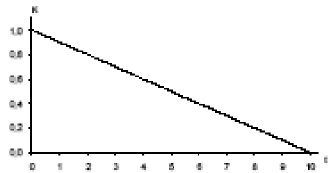
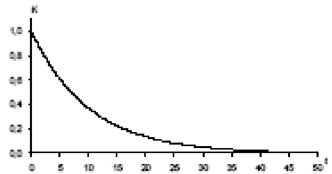
Amortizing assets

- **Definition: An amortizing asset is one that must be paid off over a specified time period, with regular payments of both principal and interest.**

- **A portfolio of amortizing assets is itself an amortizing asset**

- **Amortizing schedule**

Profile	Differential equation	Parameters	Graphic
Installment loan	$\frac{dK_t^0}{dt} = r_M K_t^0 - x$	r_M is the mortgage interest rate and x the constant payment rate	
Fixed principal loan	$\frac{dK_t^0}{dt} = -\frac{1}{T}$	T is the asset maturity	
Relative constant	$\frac{dK_t^0}{dt} = -k \cdot K_t^0$	k is the constant relative amortization rate	

Prepayment

- **Prepayment is the partial or total early redemption of the principal**
- **Prepayment is either deterministic or random**
- **The prepayment process $(Q_t)_{t \geqslant 0}$ represents the fraction of principal that has not been prepaid at date *t*.**
- **The prepayment process has the following properties:**
  - At time $t = 0$, $Q_0 = 1$
  - $(Q_t)_{t \geqslant 0}$ is a decreasing process

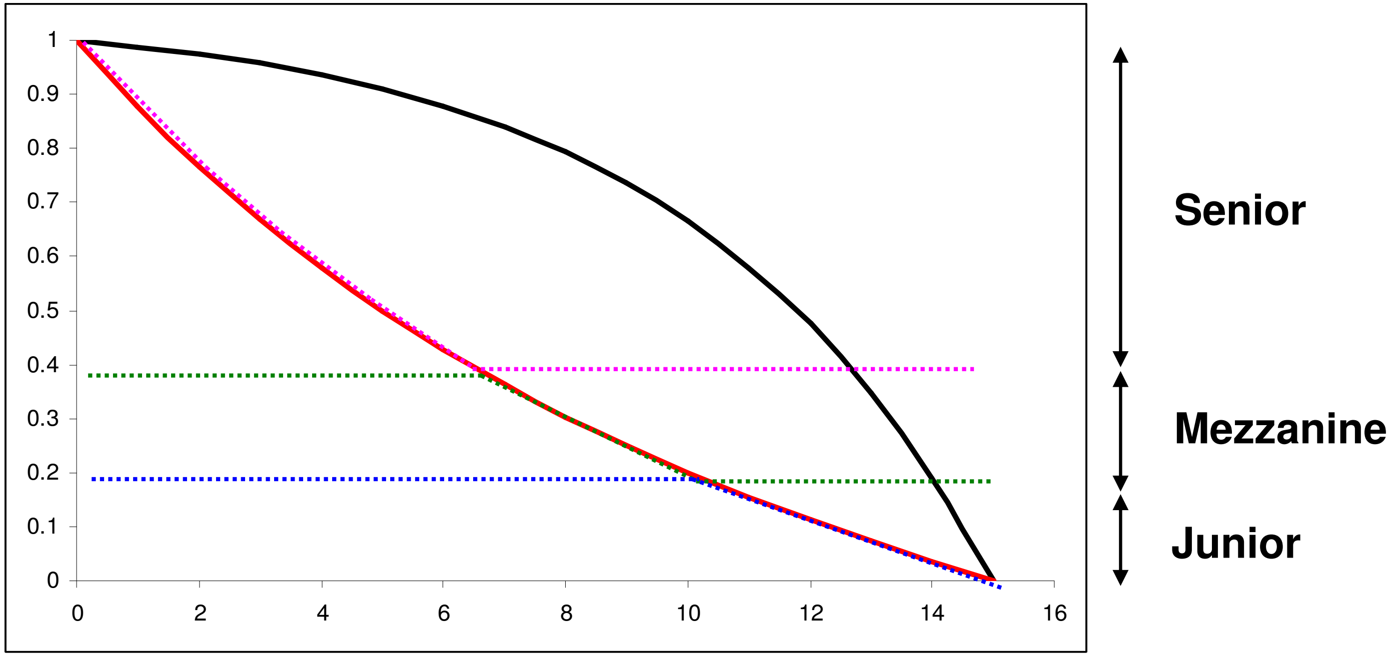
Outstanding principal amount

■ Outstanding principal amount at time *t*

$$K_t = Q_t K_t^0.$$

■ Structured exposure

$$K_t = f\left(Q_t, K_t^0\right)$$



Prepayment and measure theory

- $-dK_t$ and $-dK_t^0$ generate two probability measures
- For any measurable function $A(t)$ we define the following bracket operators

$$\langle A \rangle_0 = -\int_0^\infty A(t)\, dK_t^0 \qquad \langle A \rangle = -E\left[\int_0^\infty A(t)\, dK_t\right]$$

- Radon-Nikodym derivative

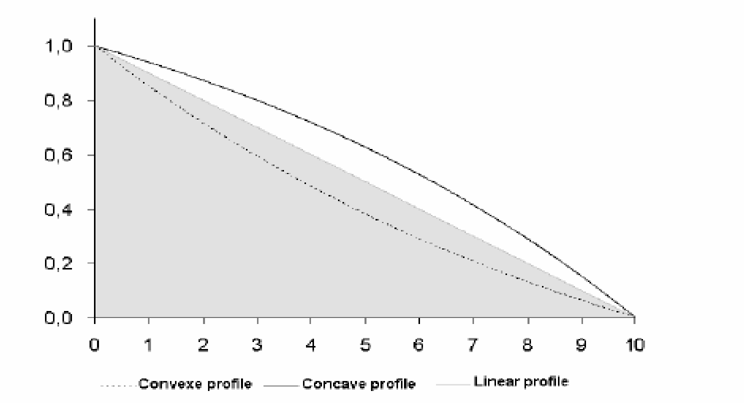
$$\langle A \rangle_0 = \langle A\, F_t \rangle$$

- Application
  - WAL $\quad WAL = -E\left[\int_0^T t\, dK_t\right] = \langle t \rangle$
  - Price $\quad P = E\left[\int_0^T e^{-yt}\left[-dK_t + y_0 K_t dt\right]\right] = \left\langle e^{-yt} \right\rangle \left(1 - \frac{y_0}{y}\right) + \frac{y_0}{y}$

Properties of the WAL

- **Integration by parts:** $WAL = E\left[\int_0^T K_t dt\right]$

- **WAL and convexity of the amortization schedule**
  - Convex $WAL \leqslant \frac{T}{2}$
  - Concave $WAL \geqslant \frac{T}{2}$

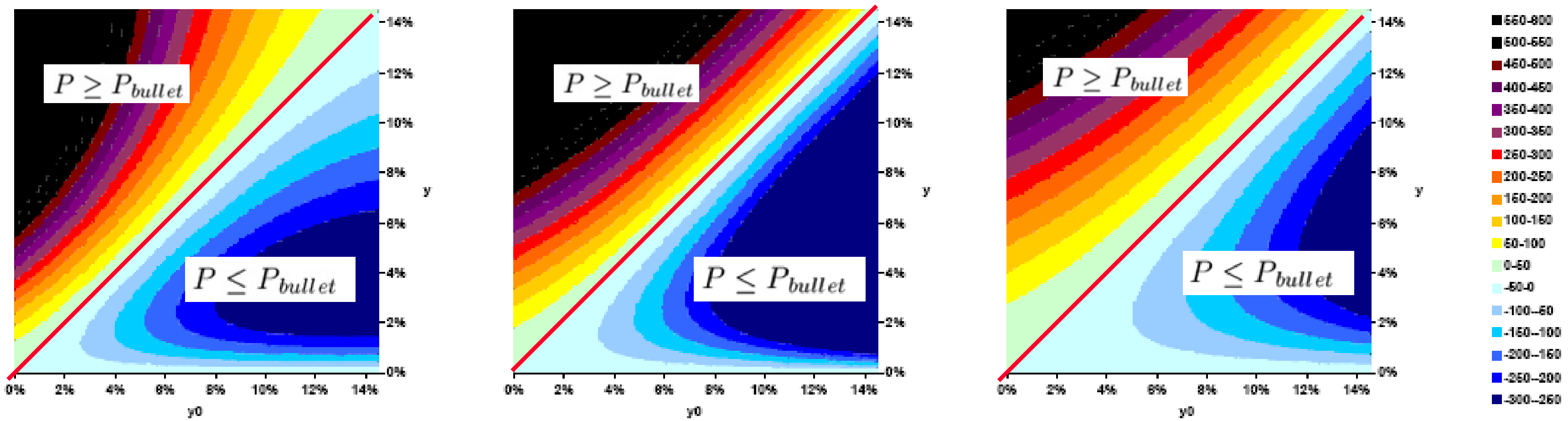


- **WAL and linearity**
  - WAL of a portfolio of amortizing assets $WAL_{pf} = \sum_i \omega_i WAL_i$
  - WAL of assets = WAL of liabilities $WAL_{asset} = \sum_j x_j WAL_j$

Price inequalities

- **Jensen's inequality** $E\left[e^X\right] \geq e^{E[X]}$ **leads to** $\langle e^{-yt} \rangle \geq e^{-y \cdot WAL}$
- **Non trivial bounds for ABS prices**

$$|P - 1| \leq |P_{bullet} - 1| = |y - y_0| \cdot \frac{1 - e^{-yWAL}}{y}$$



$T = 30,\ \lambda = 0\%$ | $T = 30,\ \lambda = 10\%$ | $T = 30,\ \lambda = 20\%$

$(P - P_{bullet})$ in bps

Constant prepayment rate and pass-through ABS

- **Constant prepayment rate** $\frac{dQ_t}{Q_t} = -\lambda dt$
- **pass-through ABS**
  - Liabilities consist in one debt tranche perfectly backed by the assets
  - Amortization schedule

$$dK_t = e^{-\lambda t} dK_t^0 - \lambda K_t dt = Q_t dK_t^0 + \frac{dQ_t}{Q_t} K_t$$

Theoretical redemptions ($e^{-\lambda t} dK_t^0$) — Early redemptions ($\lambda K_t dt$)

  - Laplace transform
    - Under the CPR assumption : $WAL(\lambda) = \int_0^{+\infty} e^{-\lambda t} K_t^0 dt$
    - Second and third moments $\frac{dWAL(\lambda)}{d\lambda} = -\frac{\langle t^2 \rangle}{2}$ and $\frac{d^2 WAL(\lambda)}{d\lambda^2} = \frac{\langle t^3 \rangle}{3}$
  - Price EDP

$$\boxed{\frac{\partial P}{\partial \lambda} = \frac{\partial P}{\partial y} - \frac{P-1}{y-y_0}}$$

$$P(\lambda, y) = 1 - (y - y_0) \cdot W(y + \lambda)$$

Model-independent approximation formulas

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Price sensitivity and convexity

- Up to second order in $y.t$

$$\frac{\partial P}{\partial y} \sim -WAL + \frac{1}{2}(2 \cdot y - y_0)\langle t^2 \rangle$$

Depends on cash-flow dispersion

- Price sensitivity approximation

$$\frac{\partial P}{\partial y} \sim \frac{(2 \cdot y - y_0)(P-1) + (y-y_0)^2 WAL}{y\,(y-y_0)} \equiv \left(\frac{\partial P}{\partial y}\right)_{approx}$$

- Convexity at first order

$$\frac{\partial^2 P}{\partial y^2} \sim \langle t^2 \rangle \sim \frac{2}{y}\left[WAL + \frac{P-1}{y-y_0}\right] \equiv \left(\frac{\partial^2 P}{\partial y^2}\right)_{approx}$$

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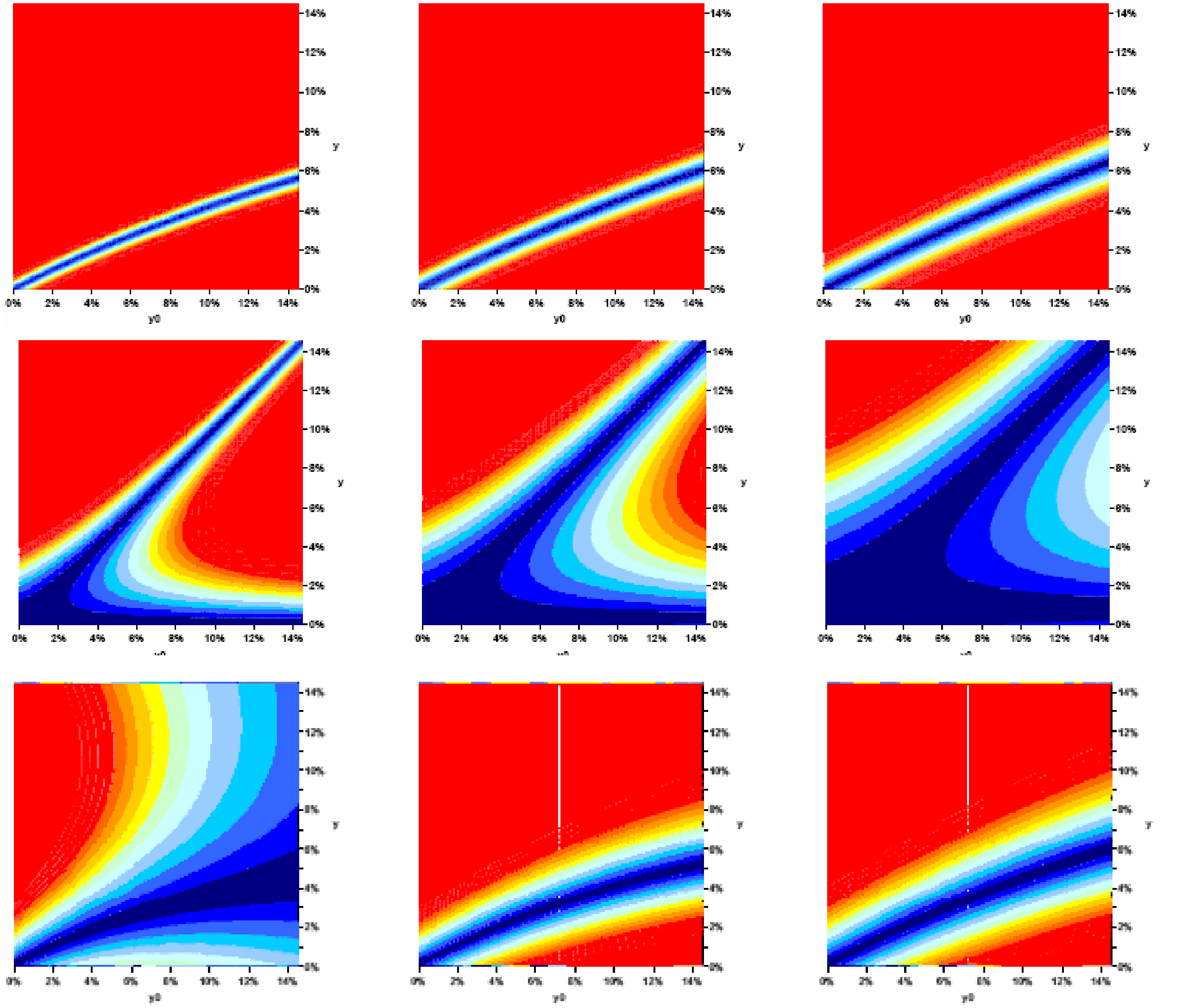
Sensitivity approximation accuracy

Sensi compared to -WAL

Sensi compared to $\left(\frac{\partial P}{\partial y}\right)_{approx}$

Sensi compared to bullet sensi

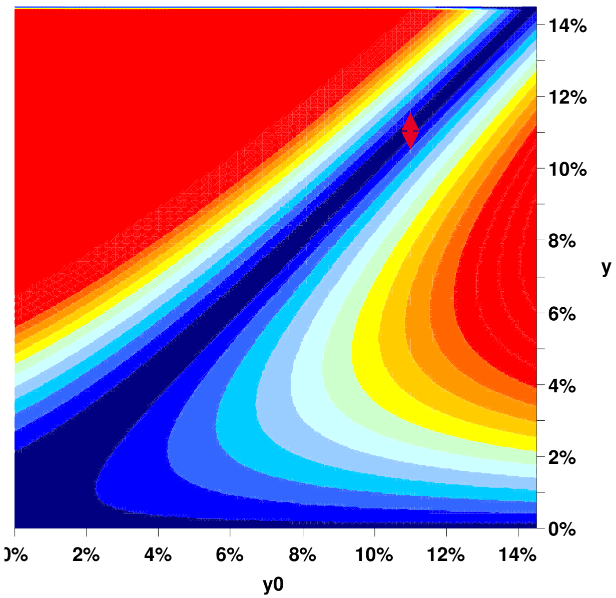


$T=30,\ \lambda=0\%$ $T=30,\ \lambda=10\%$ $T=30,\ \lambda=20\%$

Senior tranche

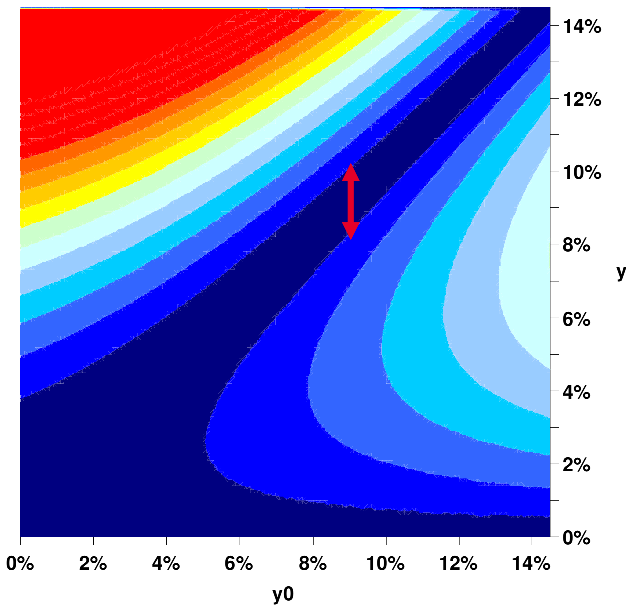
■ **Senior tranche :** $0 < A < D = 1$

- The approximation formulas will be more accurate for senior tranches than for pass-through securities



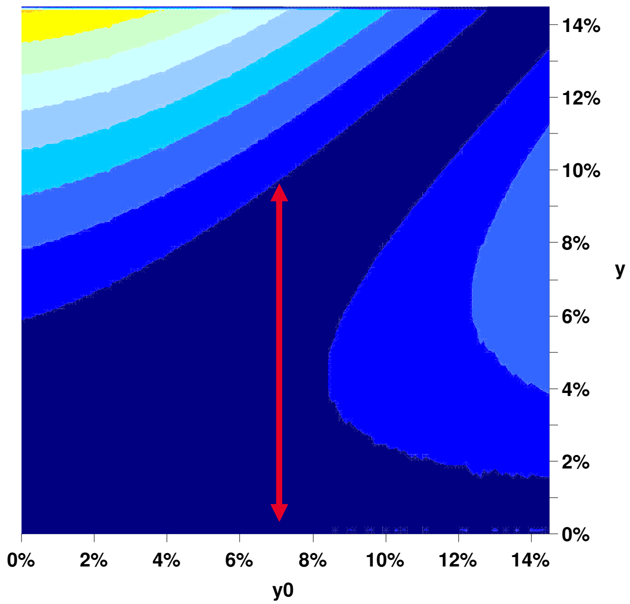
A = 0%, y_0 = 11%

$\tau_\lambda(A)$ = 30 yrs



A = 20%, y_0 = 9%

$\tau_\lambda(A)$ = 14 yrs



A = 60%, y_0 = 7%

$\tau_\lambda(A)$ = 8 yrs

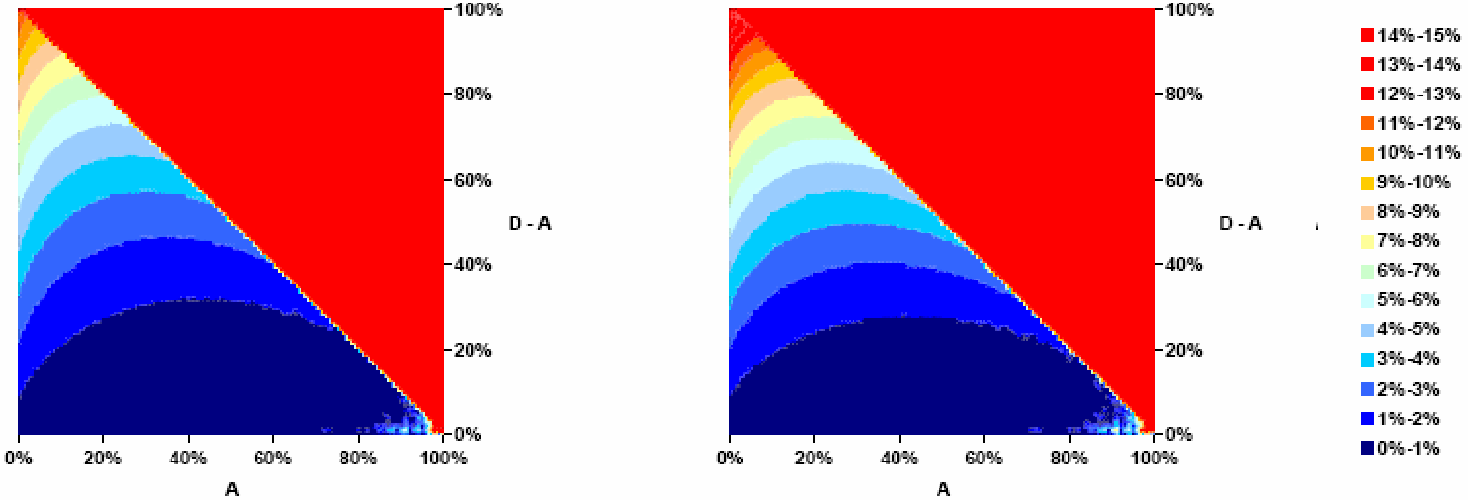
- Sensitivity to prepayment is more complex : depends on when attachement point is reached

Mezzanine tranche

■ Mezzanine tranche $0 < A < D < 1$

- **Infinitely Thin Tranches (ITT)** $D - A \to 0$
- **ITT = bullet asset with maturity equal to WAL**

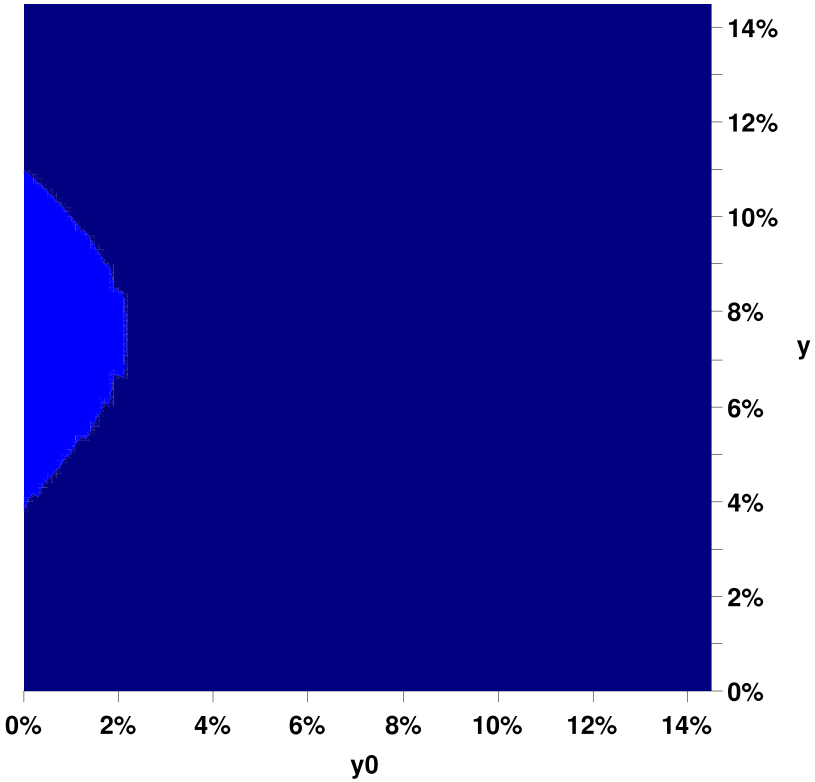
$$\frac{\partial}{\partial y} P_{bullet} = -\frac{y_0}{y^2}\left(1 - e^{-yWAL}\right) - \frac{y - y_0}{y} \cdot WAL \cdot e^{-yWAL}$$



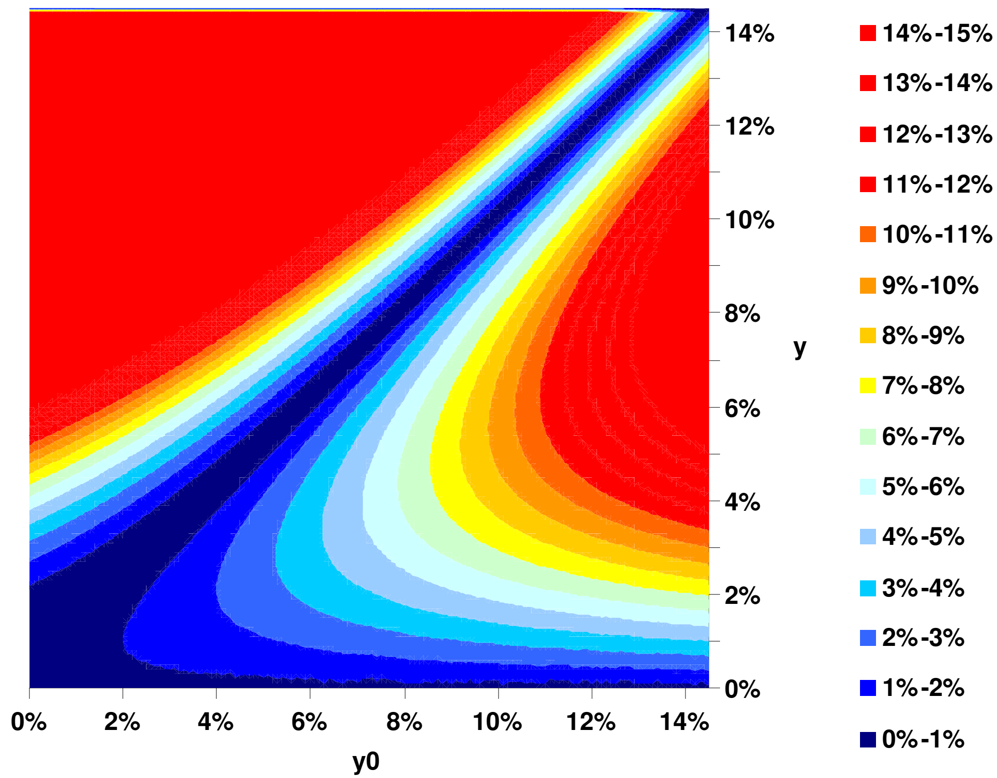
$y = y_0 = 5\%$ $y = y_0 = 7\%$

Mezzanine tranche

- **Approximation by the bullet price sensitivity is very accurate up to 20% thickness tranches**



$\frac{\partial}{\partial y} P_{bullet}$



$\left(\frac{\partial P}{\partial y}\right)_{approx}$

A = 10%, D – A = 10%, λ = 10%

Conclusion

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Conclusion

- **Prepayment impact is a change of probability measure**

- **Model-independent approximation formulas for sensitivity and convexity**
  - More accurate than the WAL or the bullet sensitivity approximations
  - Universal formulas with market data inputs: Price, WAL, Discount Margins

- **Cash-flows dispersion impact**

- **Mezzanine tranches can be easily approximated by their bullet counterparts**
  - Good accuracy up to 20% thickness