Financial Risks
New Developments in Structured Products & Credit Derivatives

On the Pricing of CDOs

Raquel M. Gaspar
Advance Research Center, ISEG
Technical University of Lisbon
Portugal

Thorsten Schmidt
Department of Mathematics
Univeristy of Leipzig
Germany

Our Setup

- Reduced-form type of model .
- Use the **general quadratic setting** to obtain closed-form solutions
  - The class of GQTS include as special cases ATS and Gaussian-QTS.
  - It is as far as we can go in terms of exponential polynomials.
- Incorporate realistic features using **shot-noise processes**:
  - Clustering of defaults within firms
  - Correlation of defaults across firms
- Deal with **portfolio credit risk derivatives**

Previous Reduced-form Models

- **Closed-form solutions** — Deterministic intensity of default
  Affine intensity models
  Gaussian Quadratic models

- **Realistic features** — Hard to replicate.
  It is hard to obtain reasonable correlation of defaults for reasonable intensity levels.

- **Other Problems** — Some affine intensity models do not guarantee positive intensity.

Our Model

The default time τ is a doubly stochastic random time with intensity

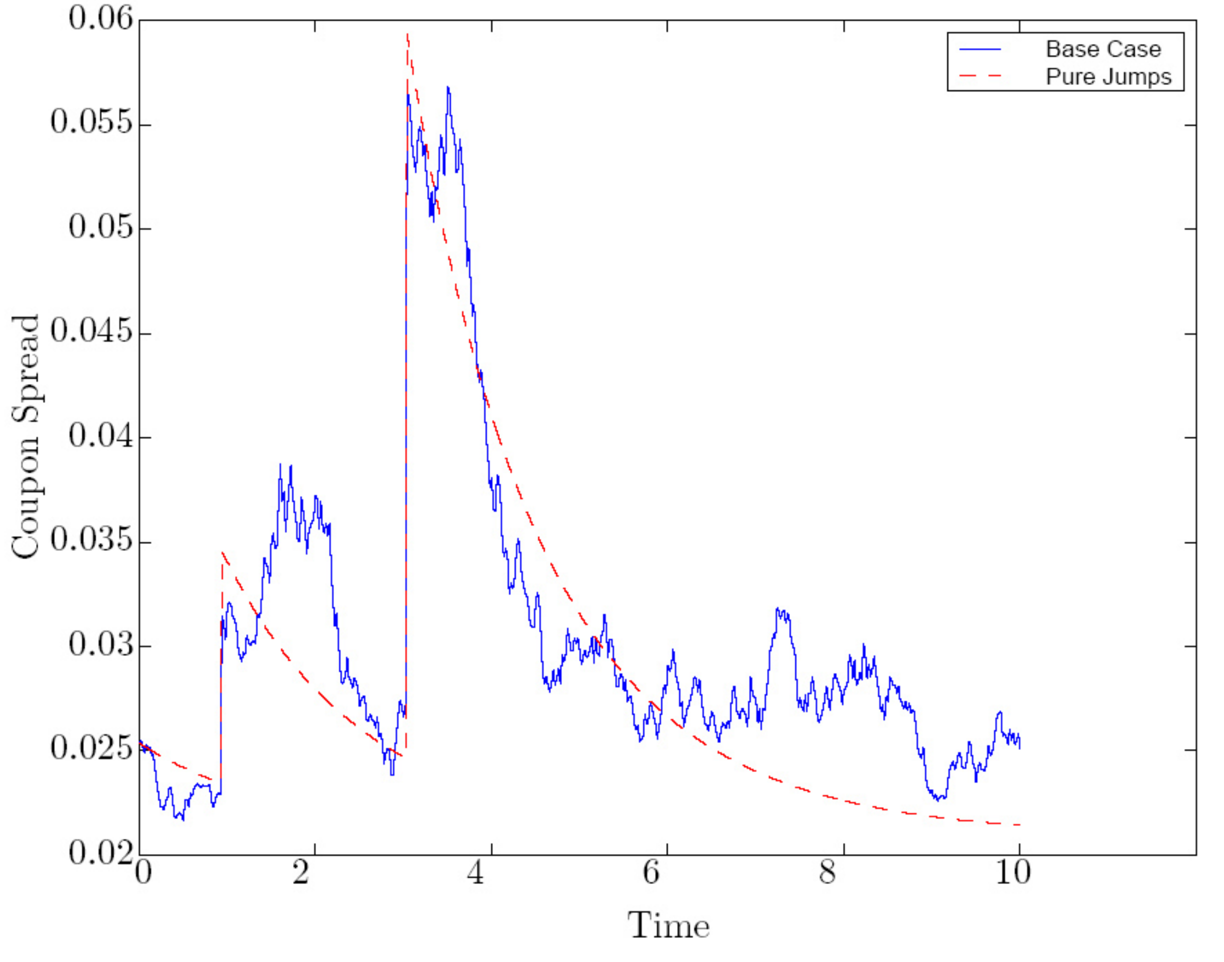
$$\mu_t = \eta_t + J_t$$

$$\eta_t = Z_t^\top \mathsf{Q}(t) Z_t + \mathsf{g}^\top(t) Z_t + \mathsf{f}(t) \qquad J_t = \sum_{\tilde{\tau}_i \leq t} Y_i h(t - \tilde{\tau}_i)$$

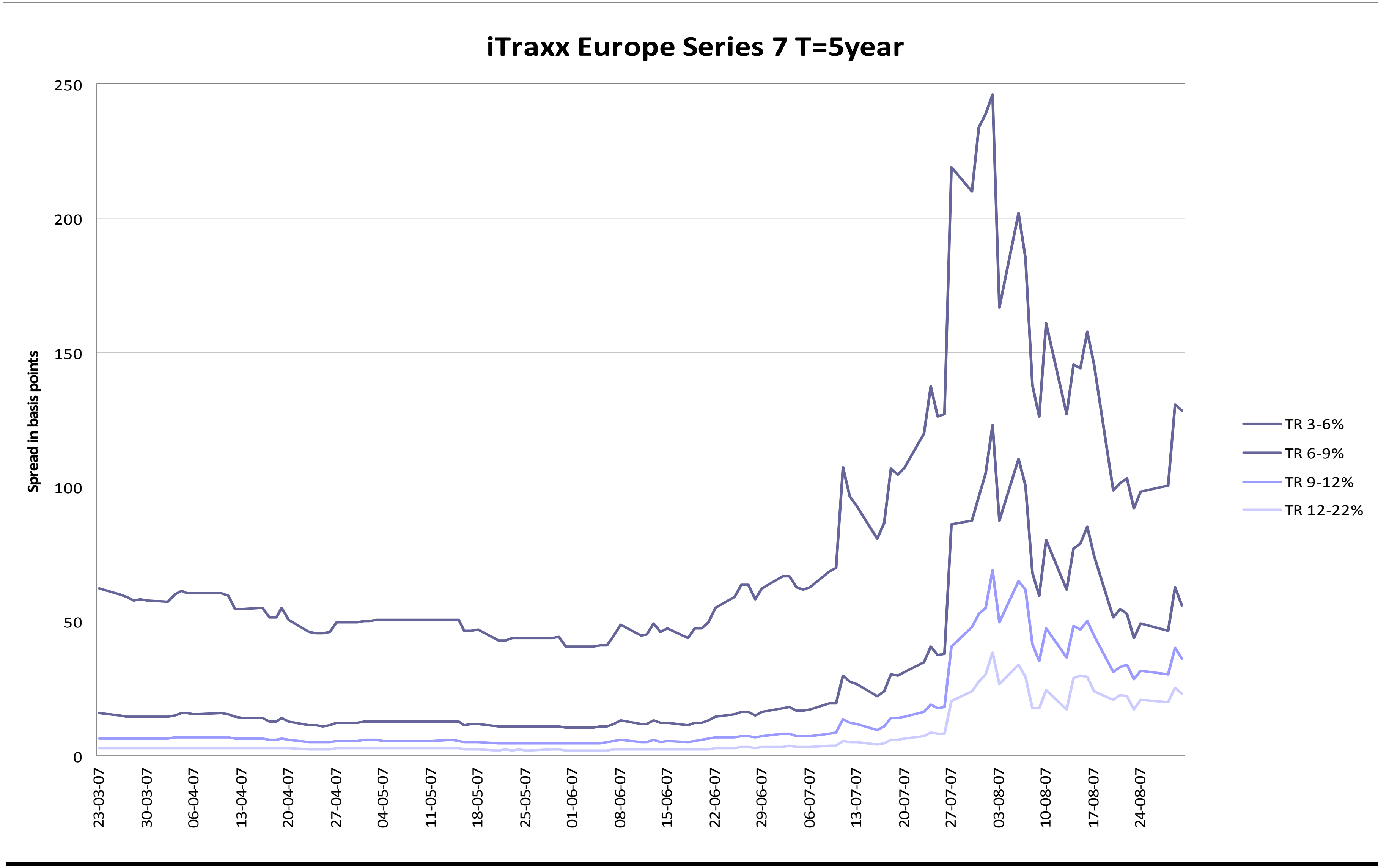
Predictable component Finite state variable driven by Wiener Process.

Unpredictable component $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$. Jump process where are the jump times of a standard Poisson process. The f $h(x) = e^{-bx}$ be, for instance,

Our intensity



How realistic is such a process ?


iTraxx Europe Series 7 T=5year
Spread in basis points
250
200
150
100
50
0
23-03-07
30-03-07
06-04-07
13-04-07
20-04-07
27-04-07
04-05-07
11-05-07
18-05-07
25-05-07
01-06-07
08-06-07
15-06-07
22-06-07
29-06-07
06-07-07
13-07-07
20-07-07
27-07-07
03-08-07
10-08-07
17-08-07
24-08-07
TR 3-6%
TR 6-9%
TR 9-12%
TR 12-22%

Advantages of our Framework

- The intensity is positive in a natural way
- We get (quasi) closed form expressions to **all** key ingredients.
- The processes η_t and J_t are **independent**.
- **Trivial to extend** to multiple firms

For each name k we have:

$$\lambda_t^k = \mu_t^k + \epsilon^k \mu_t^c$$

Each term is of quadratic + shot noise form

Firm-specific term (μ_t^k)

Systematic term ($\epsilon^k \mu_t^c$) that depends upon the sensitivity of each name to the overall economy.

- **Jump part:** Clustering/Contagion
- **Quadratic part:** Business Cycle

A concrete instance

$$\mu_t^k = \eta_t^k = \left(Z_t^k\right)^2 \quad \begin{array}{rcl} dZ_t^i &=& \left[\beta_i(t) - \alpha_i Z_t^i\right] dt + \sigma_i dW_t^i, \quad i = 1,2 \\ dr_t &=& \alpha_r \left[\beta_r - r_t\right] dt + \sigma_r \sqrt{r_t} dW_t^r \end{array}$$

$$\mu^c = J^c + \delta r \quad \begin{array}{l} J_t^c = \sum_{\tilde{\tau}_i < t} Y_i h(t - \tilde{\tau}_i) \;\; Y_i \sim \chi^2(2) \quad h(t) = e^{-bt}, \; b > 0 \\ \tilde{\tau}_i \text{ are jumps of a Poisson process with intensity } l^c \end{array}$$

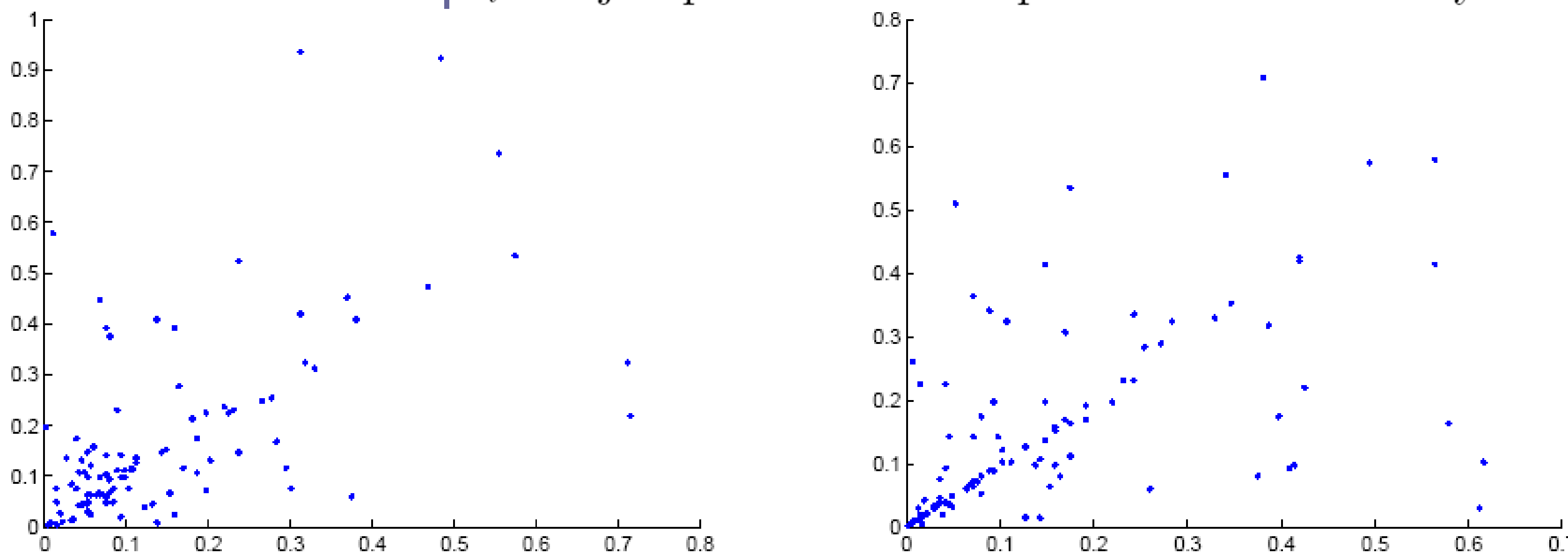


Figure 3: Simulated defaults of two companies according to the concrete model. Parameters are for $i = 1,2$: $\beta_i = 1, \alpha_i = 0.5, \sigma_i = 0.2, l^c = 2, b = 0.5$. $Y_i \sim \chi^2(2)$ and $r = 0$. The left picture if for $\epsilon_i = 0.1$, the right $\epsilon_i = 0.5$.

Default Correlation

$$\mu_t^k = \eta_t^k = \left(Z_t^k\right)^2 \quad \Bigg| \quad \begin{aligned} dZ_t^i &= \left[\beta_i(t) - \alpha_i Z_t^i\right] dt + \sigma_i dW_t^i, \quad i = 1,2 \\ dr_t &= \alpha_r \left[\beta_r - r_t\right] dt + \sigma_r \sqrt{r_t} dW_t^r \end{aligned}$$

$$\mu^c = J^c + \delta r \quad \Bigg| \quad J_t^c = \sum_{\tilde{\tau}_i < t} Y_i h(t - \tilde{\tau}_i) \quad Y_i \sim \chi^2(2) \;\; h(t) = e^{-bt}, \;\; b > 0$$

$\tilde{\tau}_i$ are jumps of a Poisson process with intensity l^c

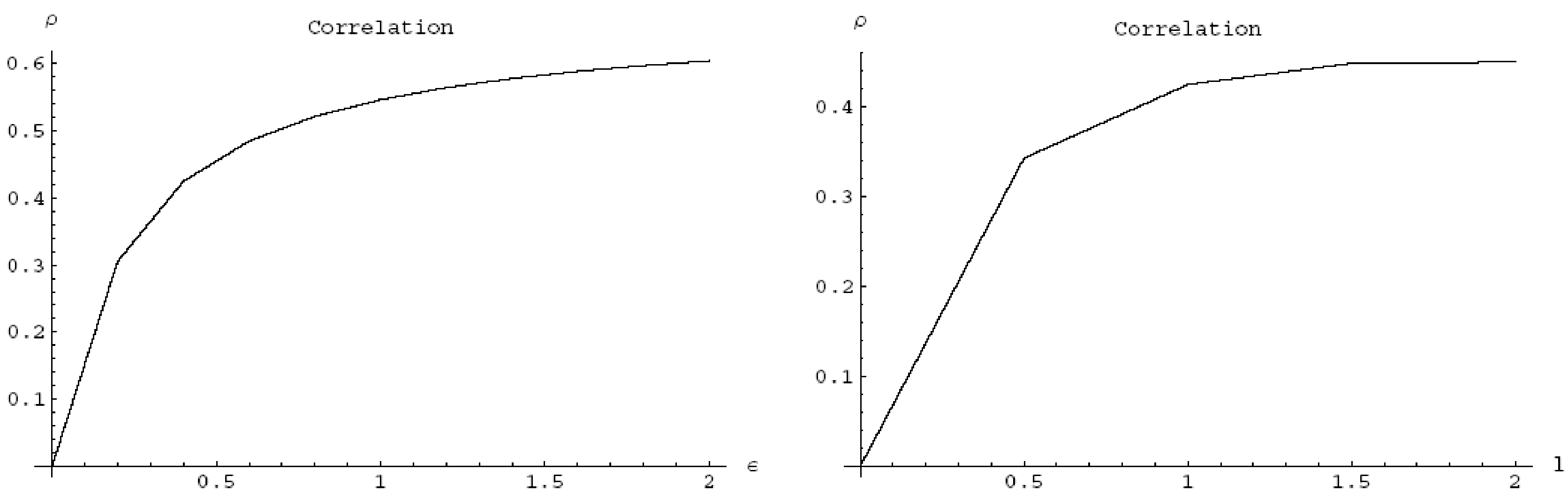


Figure 4: Model parameters:$\alpha = 0.5$, $\beta = 0.1\alpha$, $\sigma = 0.1$,$b = 0.5$. The Graph shows the correlation for varying $\epsilon = \epsilon_1 = \epsilon_2$ (left, $l^c = 1$) and $l = l^c$ (right, $= \epsilon_1 = \epsilon_2 = 0.4$).

Key Ingredients

We consider the following risk-neutral expectations the key ingredients in credit risk reduced-form models

$$S_\eta^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\eta_s^k ds} | \mathbf{F}_t^W\right]$$

$$\Gamma_\eta^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[\theta\eta_T^k e^{-\int_t^T \theta\eta_s^k ds} | \mathbf{F}_t^W\right]$$

$$\bar{S}_\eta^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\eta_s^k ds} | \mathbf{F}_t^W\right]$$

$$S_J^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta J_s^k ds} | \mathbf{F}_t^J\right]$$

$$\Gamma_J^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[\theta J_T^k e^{-\int_t^T \theta J_s^k ds} | \mathbf{F}_t^J\right]$$

$$\bar{\Gamma}^k(\theta,t,T) := \mathbb{E}^{\mathbb{Q}}\left[\theta\eta_T^k e^{-\int_t^T r_s+\theta\eta_s^k ds} | \mathbf{F}_t^W\right]$$

→ **All** can be obtained either in **explicit form** or up to the **solution of a ODE system**

$$
\begin{aligned}
S^k(\theta,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \theta\mu_s^k ds}|\mathbf{F}_t^W\right] = S_\eta^k(\theta,t,T)\cdot S_J^k(\theta,t,T) \\
\bar{S}^k(\theta,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s+\theta\mu_s^k ds}|\mathbf{F}_t^W\right] = \bar{S}_\eta^k(\theta,t,T)\cdot S_J^k(\theta,t,T) \\
\Gamma^k(\theta,t,T) &:= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu^k e^{-\int_t^T \theta\mu_s^k ds}|\mathbf{F}_t^W\right] \\
&= \Gamma_\eta^k(\theta,t,T)S_J^k(\theta,t,T)+\Gamma_J^k(\theta,t,T)S_\eta^k(\theta,t,T) \\
\bar{\Gamma}^k(\theta,t,T) &= \mathbb{E}^{\mathbb{Q}}\left[\theta\mu^k e^{-\int_t^T r_s+\theta\mu_s^k ds}|\mathbf{F}_t^W\right] \\
&= \bar{\Gamma}_\eta^k(\theta,t,T)S_J^k(\theta,t,T)+\Gamma_J^k(\theta,t,T)\bar{S}_\eta^k(\theta,t,T).
\end{aligned}
$$

What can we obtain in (quasi) closed form using these key ingredients?

A lot…

For each name

- Survival (and default) probabilities: $\mathbb{Q}_S^k(t, t_n)$
- Prices of defaultable discount bonds: $\bar{p}_0^k(t, t_n)$
- Prices of digitals : $e^{*k}(t, t_{n-1}, t_n)$

(price of a payoff of 1 if default occurs in $(t_{n-1}, t_n]$)

For any two names

- Default correlation: $\rho^{i,j}(t, T)$

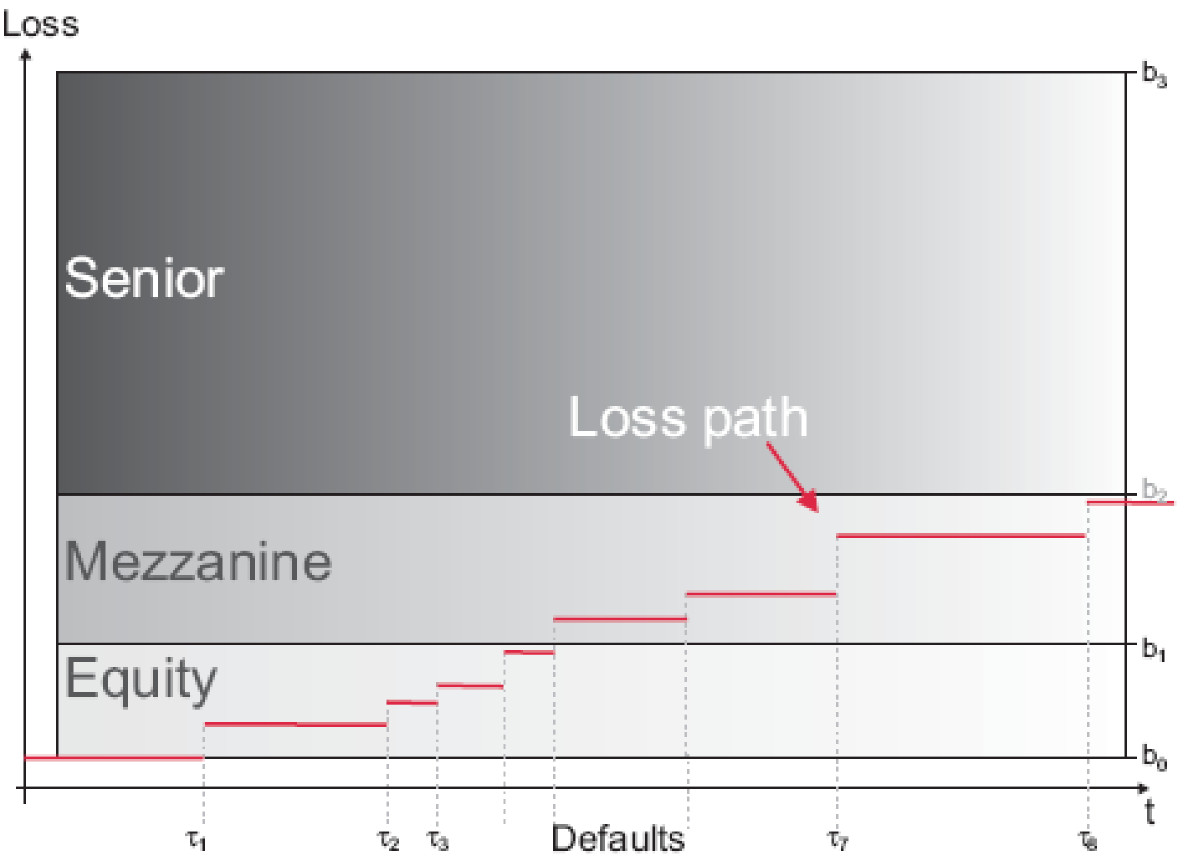
For portfolios

- First-to-default spreads: $s^{FD}(t)$
- Total loss distributions under $\mathbb{Q}(L_T \leq x | \mathbf{G}_t)$ $\mathbb{Q}^T(L_T \leq x | \mathbf{G}_t)$
- Par spreads for CDO tranches on credit indices: $S(t, b_1, b_2)$

CDOs

The loss process:

$$L(t) := \sum_{\tau^k \leq t} q^k M^k$$



Tranche Losses:

$$L^i(t) = \begin{cases} 0 & \text{if} \quad L(t) < b^{i-1} \\ L(t) - b^{i-1} & \text{if} \quad b^{i-1} \leq L(t) < b^i \\ b^i - b^{i-1} & \text{if} \quad L(t) \geq b^i \end{cases}$$

Credit Indices

- The iTraxx is effectively a **portfolio of** 125 **single CDS**.
- To guarantee liquidity, the portfolio is reorganized (the so-called series) semiannually.
- The aim of this procedure is to guarantee that the underlying portfolio stays in a certain class of credit worthiness.

Mathematical Setting:

- Assume the notional is 1
- The credit index is on K names, each represented by a CDS with spread
- All names are in the same credit class so it reasonable to assume some **homogeneity** in terms of losses (given default), sensitivity to common factors and tenor structure

$$q^k = q \qquad \epsilon^k = \epsilon \qquad t_j = j\Delta, 1 \leq j \leq N^*$$

- Each name have equal weight ($M^k = M$)
- Typically, the recovery in traded indices is set to zero, but we stay a bit more general at this point.

Index Spread:

The payment stream of the credit index is as follows.

$$S\Delta \frac{K - N_{t_n}}{K}$$

Fixed leg → Defaulting leg

Fixed leg ← Defaulting leg

$$\sum_{\tau^k \in (t_{n-1}, t_n]} (1-q) = (1-q)(N_{t_n} - N_{t_{n-1}})$$

- The spread of a credit index is and average of single CDS spreads **only until** the first default occurs.

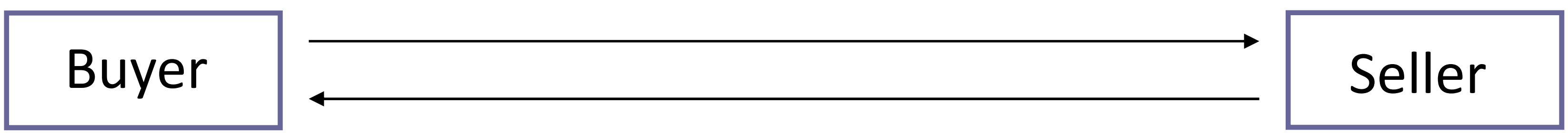
- The exact expression of the index spread is given by

$$S_t = \bar{q}K \frac{\sum_{n=2}^{N^*} \mathbf{1}_{\{t_n \geq t\}} \sum_{k=1}^{K} e^{*k}(t, t_{n-1}, t_n)}{\Delta \sum_{t_n \geq t}^{t_{N^*-1}} \sum_{k=1}^{K} \bar{p}_0^k(t, t_n)}$$

Tranches on Credit Indices

- We consider the overall nominal to be 1.
- A **tranche** refers to an interval (b1; b2] [0; 1].
- Investing (selling protection) in a tranche is again done by a swap where the following payments are exchanged:

$$S\left(\mathbf{1}_{\{L_{t_n}\leq b_1\}}+\frac{(b_2-L_{t_n})^+}{b_2-b_1}\mathbf{1}_{\{L_{t_n}>x_1\}}\right)$$



$$\left(\left(b_2-L_{t_{n-1}}\right)^+-\left(b_2-L_{t_n}\right)^+\right)\mathbf{1}_{\{L_{t_n}>b_1\}}$$

- The par spread of tranche (b1; b2] is given by

$$S(t,b_1,b_2) =$$

$$= \frac{\int_{b_1}^{b_2} \left(p(t,t_1)\mathbf{1}_{\{L_t<y\}} - C(t,t_{N^*},y) + \sum_{n=2}^{N^*-1} C(t,t_n,y)\cdot\left(1-\frac{p(t,t_{n+1})}{p(t,t_n)}\right)\right) dy}{\sum_{n=1}^{N^*-1} \frac{1}{b_2-b_1}\int_{b_1}^{b_2} C(t,t_n,y)\,dy}$$

where

$$C(t,T,y) = p(t,T)\mathbb{Q}^T\left(L_T < y \middle| \mathbf{G}_t\right)$$

⇩

Thus also in (quasi) closed-form !

Conclusion

- We propose to use a **class of quadratic shot-noise reduced form models** when dealing with credit risk.

- We show that this class of models:
  - is flexible enough o capture realistic **correlation and clustering** of defaults
  - it is rather **parsimonious** as all (default) correlations result from a common term in the default intensities of the names
  - allow to derive in (quasi) **close form solutions** to all interesting variables of credit risk. Even in the case of portfolio credit products
  - **consistently** prices single-name and portfolio credit products.

References

- **This paper**:
  - Gaspar, R.M. and T. Schmidt (2008) On the Pricing of CDOs, in *Credit Derivatives* (Ed. P.U. Ali and G. Gregouriou), McGraw-Hill.
- **Related papers:**
  - Gaspar, R.M. and T. Schmidt (2007), Term Structure Models with shot-noise effects, *available at SSRN and DefaultRisk.com*
  - Gaspar, R.M. (2004), "General Quadratic Term Structures of Bond, Futures and Forward Prices", SSE/EFI working papers Series in Economics and Finance, n.559.

Further Research

- Calibrate a concrete instance of our class of models to market data!

Other Suggestions Welcome…