Default contagion in large homogeneous portfolios

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Financial Risks: New Developments in Structured Products and Credit Derivatives
Paris

March 27, 2008

Content

- Introduce default contagion in an intensity based credit risk model.
- Present expressions for multivariate default and survival distributions, both for ordered and unordered default times.
- Present formulas for CDO tranche spreads in our model.
- Show some numerical results implied by market data from the calibrated model.

Research for portfolio credit models

- Exists huge amount of research in portfolio credit risk
- Our contribution:
  - Intensity based model where dynamic default dependencies among obligors are expressed in an intuitive, direct and compact way, by using default contagion.
  - Enables fast computationally tractable closed-form expressions for multivariate default and survival distributions, credit derivatives, and much more.....
- Default contagion models clustering of defaults. It has been treated in many recent articles.
- Our approach close to papers by for example Davis et al, Frey et al., Jeanblanc et al., Laurent et al.

Default contagion in intensity models

The general case (i.e. inhomogeneous portfolio): for default times $\tau_1, \tau_2 \ldots, \tau_m$, define $N_{t,i} = 1_{\{\tau_i \leq t\}}$ and $\mathcal{F}_{t,i} = \sigma\left(N_{s,i}; s \leq t\right)$, $\mathcal{F}_t = \bigvee_{i=1}^{m} \mathcal{F}_{t,i}$.

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$ given by

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad \tau_i \geq t,$$

where $a_i \geq 0$ and $b_{i,j}$ such that $\lambda_{t,i} \geq 0$. Note that $\lambda_{t,i} = 0$, if $\tau_i \leq t$.

Intensity for obligor i jumps by an amount $b_{i,j}$ at default of obligor j

- $b_{i,j} > 0$ means that i is put at higher risk by the default of j.
- $b_{i,j} < 0$ means that i benefits from the default of j.
- $b_{i,j} = 0$ means that i is unaffected by the default of j.

Intensity based models

- The intensities $\{\lambda_{t,i}\}$ uniquely determines the multivariate distribution for $\tau_1, \tau_2 \ldots, \tau_m$
- Not obvious how to go from $\{\lambda_{t,i}\}$ to distribution of $\{\tau_i\}$
- Solution: Reformulate as a Markov jump process.
- In a nonsymmetric portfolio, only practical for m up to 20, say.
- If portfolio homogeneous, the Markov approach works for large portfolios (CDO's). Then $m = 125$ is no problem.
- Symmetry implies that $\lambda_{t,i} = \lambda_t$, for $\tau_i > t$, where

$$\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}} \tag{1}$$

where $\{T_k\}$ ordering of $\{\tau_i\}$. Recall that $\lambda_{t,i} = 0$, for $\tau_i \leq t$.

Translate into Markov jump process

Proposition 1

There exists a Markov jump process $(Y_t)_{t\geq 0}$ on a finite state space $\mathbf{E} = \{0, 1, 2, \dots, m\}$, such that the stopping times

$$T_k = \inf\{t > 0 : Y_t = k\}, \quad k = 1, \dots, m$$

are the ordering of m exchangeable stopping times $\tau_1, \dots, \tau_m$ with intensities $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \leq t\}}$. The generator $\mathbf{Q}$ to Y_t is given by

$$\mathbf{Q}_{k,k+1} = (m-k)\left(a + \sum_{j=1}^{k} b_j\right) \quad \text{and} \quad \mathbf{Q}_{k,k} = -\mathbf{Q}_{k,k+1} \quad \text{for} \quad k = 0, 1, \dots, m-1$$

where the other entries in $\mathbf{Q}$ are zero. The Markov process starts in $\{0\}$ so the initial distribution is given by $\boldsymbol{\alpha} = (1, 0, 0, \dots, 0)$.

Due to **Proposition 1**, we can use matrix-analytic methods to find compact, computationally tractable closed-form expressions for many quantities.

Example: multivariate distributions

Consider m obligors with default intensities (1).

Proposition 2

Let $k_1 < \ldots < k_q$ be an increasing subsequence in $\{1, \ldots, m\}$ where $1 \leq q \leq m$. Furthermore, let $t_1 < t_2 < \ldots < t_q$. Then,

$$\mathbb{P}\left[T_{k_1} \leq t_1, \ldots, T_{k_q} \leq t_q\right] = \boldsymbol{\alpha}\left(\prod_{i=1}^{q} e^{\mathbf{Q}(t_i - t_{i-1})}\mathbf{N}_{k_i}\right)\mathbf{1}$$

where $\mathbf{N}_k$ is $(m+1)\times(m+1)$ diagonal matrix, $(\mathbf{N}_k)_{j,j} = 1_{\{j \geq k\}}$

Proposition 3

Let $t_1 < t_2$. Then,

$$\mathbb{P}\left[\tau_1 \leq t_1, \tau_2 \leq t_2\right] = \frac{(m-2)!}{m!}\left(\alpha e^{\mathbf{Q}t_1}\mathbf{n} + \sum_{k_1=1}^{m}\sum_{k_2=k_1+1}^{m}\alpha e^{\mathbf{Q}t_1}\mathbf{N}_{k_1}e^{\mathbf{Q}(t_2-t_1)}\mathbf{N}_{k_2}\mathbf{1}\right)$$

where $\mathbf{n}$ is column vectors in $\mathbb{R}^{m+1}$ such that $\mathbf{n}_j = \frac{j(j-1)}{2}$.

Other examples

Proposition 4

Let q be a integer where $1 \leq q \leq m$ and let $t > 0$. Then,

$$\mathbb{P}\left[\tau_1 \leq t, \ldots, \tau_q \leq t\right] = \alpha e^{\mathbf{Q}t}\mathbf{d}^{(q)} \quad \text{where} \quad \mathbf{d}_j^{(q)} = \frac{\binom{j}{q}}{\binom{m}{q}} 1_{\{j \geq q\}}.$$

By using similar techniques as in the previous propositions, we can also find

- $\mathbb{P}\left[T_{k_1} > t_1, \ldots, T_{k_q} > t_q\right]$,
- $\mathbb{P}\left[\tau_1 > t_1, \tau_2 > t_2\right]$
- $\mathbb{P}\left[\tau_1 > t, \ldots, \tau_q > t\right]$
- $\mathbb{E}\left[T_k\right]$ and $\text{Corr}(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}})$ and $\mathbb{E}\left[\tau_1\right]$
- Formulas for credit derivatives, such as CDS and CDO tranche spreads.

The CDO tranche spreads

- Consider a synthetic CDO consisting of m credit default swaps on obligors with default times $\tau_1, \tau_2 \dots, \tau_m$ and losses $\ell_1, \ell_2, \dots, \ell_m$.

- The credit loss L_t for this portfolio at time t is $L_t = \sum_{i=1}^{m} \ell_i 1_{\{\tau_i \leq t\}}$.

- The $[a, b]$-tranche loss is $L_t^{(a,b)} = (L_t - a) 1_{\{L_t \in [a,b]\}} + (b - a) 1_{\{L_t > b\}}$.

- The CDO tranche spread $S_{(a,b)}(T)$ for tranche $[a, b]$ up to time T is

$$S_{(a,b)}(T) = \frac{B_T \mathbb{E}\left[L_T^{(a,b)}\right] + \int_0^T r_t B_t \mathbb{E}\left[L_t^{(a,b)}\right] dt}{\sum_{n=1}^{4T} B_{t_n} \left(b - a - \mathbb{E}\left[L_{t_n}^{(a,b)}\right]\right) \frac{1}{4}} \qquad \text{if } a > 0$$

where r_t deterministic, $B_t = \exp\left(-\int_0^t r_s ds\right)$ and $t_n = \frac{n}{4}$.

The CDO tranche spreads in the model

Proposition 5

Consider a synthetic CDO on a portfolio with m obligors that satisfy (1) and assume that the interest rate r is constant. Then,

$$S_{(a,b)}(T) = \frac{\left(\boldsymbol{\alpha} e^{\mathbf{Q}T} e^{-rT} + \boldsymbol{\alpha}\mathbf{R}(0,T)r\right)\boldsymbol{\ell}^{(a,b)}}{\sum_{n=1}^{4T} e^{-rt_n}\left(b - a - \boldsymbol{\alpha} e^{\mathbf{Q}t_n}\boldsymbol{\ell}^{(a,b)}\right)\frac{1}{4}} \qquad \text{if} \quad a > 0$$

for $t_n = \frac{n}{4}$, where

$$\mathbf{R}(0,T) = \int_0^T e^{(\mathbf{Q}-r\mathbf{I})t} dt = \left(e^{\mathbf{Q}T} e^{-rT} - \mathbf{I}\right)\left(\mathbf{Q} - r\mathbf{I}\right)^{-1}$$

and $\boldsymbol{\ell}^{(a,b)}$ is a column vector in $\mathbb{R}^{m+1}$, defined by

$$\ell_k^{(a,b)} = \begin{cases} 0 & \text{if} \quad k < \lceil am/\ell \rceil \\ k\ell/m - a & \text{if} \quad \lceil am/\ell \rceil \leq k \leq \lfloor bm/\ell \rfloor \\ b - a & \text{if} \quad k > \lfloor bm/\ell \rfloor \end{cases}$$

Calibrating the model

- Let $\mathbf{a} = (a, b_1, b_2, \ldots, b_{m-1})$ denote the m parameters describing λ_t.
- Let $\{C_j(T; \mathbf{a})\}$ be the model spreads for the instruments used in the calibration (=CDO tranche spreads, index CDS spread and average CDS spread) and $\{C_{j,M}(T)\}$ are the corresponding market spreads.
- We have 7 instruments: 5 tranches, the index and the average CDS.
- The vector $\mathbf{a}$ is obtained as $\mathbf{a} = \underset{\hat{\mathbf{a}}}{\operatorname{argmin}} \sum_{j=1}^{7} \left(C_j(T; \hat{\mathbf{a}}) - C_{j,M}(T)\right)^2$.
- To reduce the number of unknown parameters in $\mathbf{a}$ we assume that

$$b_k = \begin{cases} b^{(1)} & \text{if} \quad 1 \leq k < \mu_1 \\ b^{(2)} & \text{if} \quad \mu_1 \leq k < \mu_2 \\ \vdots & \\ b^{(6)} & \text{if} \quad \mu_5 \leq k < \mu_6 = m \end{cases}$$

where $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_4, \mu_6\} = \{7, 13, 19, 25, 46, 125\}$.

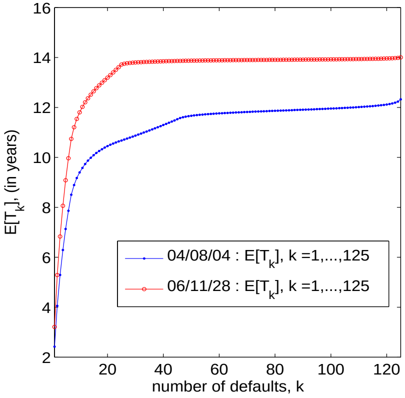
Good calibration for $T = 5$

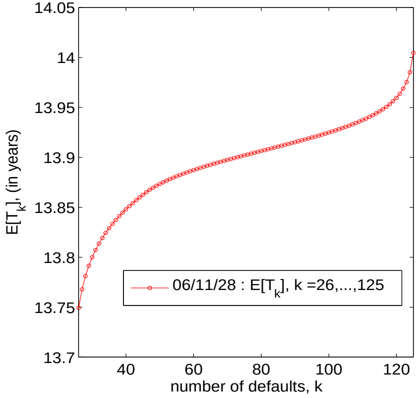
iTraxx Europe Series. Left: August 4^{th}, 2004. Right: November 28^{th}, 2006. The market and model spreads and the corresponding absolute errors (in bp). The $[0, 3]$ spread is quoted in %. Maturities are for five years, $r = 3\%, \ell = 60\%$.

	Market	Model	error (bp)	Market	Model	error (bp)
$[0, 3]$	27.6	27.6	0.0000385	14.5	14.5	0.008273
$[3, 6]$	168	168	0.000316	62.5	62.48	0.02224
$[6, 9]$	70	70	0.000498	18	18.07	0.07275
$[9, 12]$	43	43	0.0005563	7	6.872	0.1282
$[12, 22]$	20	20	0.0004006	3	3.417	0.4169
index	42	42.02	0.01853	26	26.15	0.1464
avg CDS	42	41.98	0.01884	26.87	26.13	0.7396
Σ abs.cal.err			0.03918 bp			1.534 bp

Market imply extreme clustering

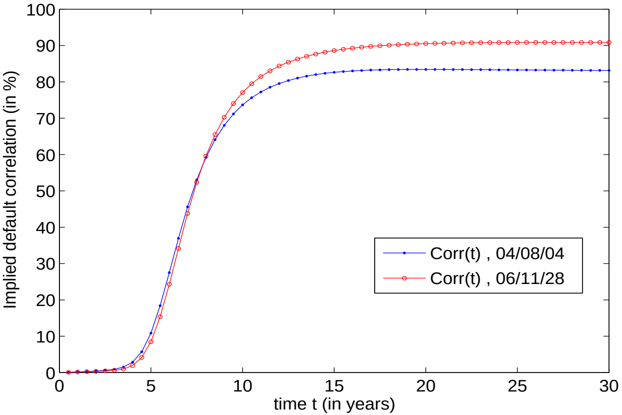
The implied expected ordered default times $\mathbb{E}[T_k]$ for the 2004-08-04 and 2006-11-28 portfolios (under risk-neutral measure).





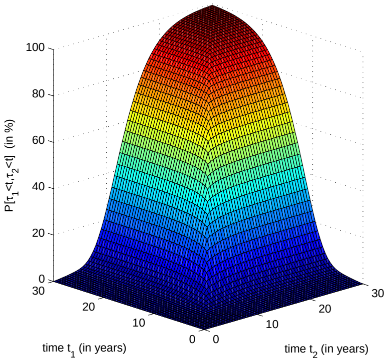
The implied default correlation

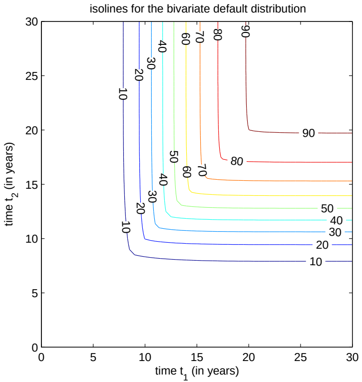
The implied default correlation $\rho(t) = \text{Corr}(1_{\{\tau_i \le t\}}, 1_{\{\tau_j \le t\}})$, $i \neq j$ as function of time for the 2004-08-04 and 2006-11-28 portfolios.



The implied bivariate default distribution

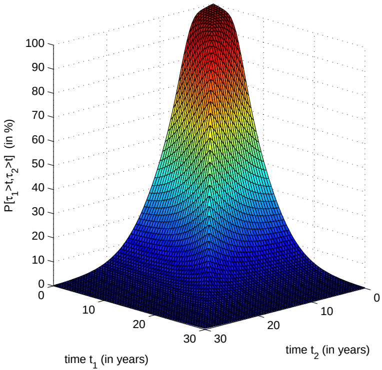
The implied bivariate default distribution (left) and its isolines (right), 06-11-28.

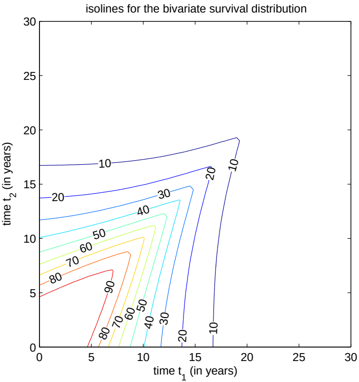




The implied bivariate survival distribution

The implied bivariate survival distribution (left) and its isolines (right), 06-11-28.





Thank you for your attention!