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Pricing of Defaultable Securities in a Multi-Factor Quadratic Gaussian Model

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Abstract

We present the multi-factor quadratic reduced form model for pricing of credit risky securities. We use quadratic Gaussian processes to model the short term interest rate and the intensity of default showing that we get tractable formulas for the price of credit default swaps and credit default swaptions.

1 Introduction

There is some work in the pricing of credit default swaps and options on credit default swaps. In Brigo & Alfonsi (2004) an extension of the Cox-Ingersoll-Ross(CIR) model known as CIR++ is used to model the short term interest rate and the intensity of default. When the interest rate and the intensity of default are independent, closed form formulas for the price of single name credit default swaps are provided. Moreover the independence enables the separation of the calibration of the short term interest rate to caps or swaptions from the calibration of the intensity of default to quotes of credit default swaps. However when there is correlation between the interest rate and intensity of default, the CIR++ reduced form model does not enable the calculation of the price of credit default swaps in closed form, therefore

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the calibration of the model has to be done through the use of Monte Carlo simulation or through a Gaussian dependence mapping. In the reduced model adopted in this paper, the use of quadratic Gaussian processes enables us to calculate the price of credit default swaps in a closed form so that the calibration of the model to the default term structure and quotes of credit default swaps can be done through analytic formulas. The assumption of a correlation between the interest rate and the intensity of default does not prevent us from obtaining the semi-analytic formulas. In the first section, we present a detail explanation of a reduced form model based on quadratic Gaussian processes and how we can calibrate such a model to the default free and defaultable term structures. The second section provides details of the pricing formulas for credit default swaps and credit default swaptions.

2 Pricing of Credit Default Swaps

In this section we first show how we can calculate the price of a domestic credit default swap in a closed form under the quadratic Gaussian factor model. We first extend the procedure to extract the probability of default from quotes of credit default swaps in a reduced form model of credit risk where the intensity of default is deterministic given in Martin et al. (2001) to a reduced form model of credit risk where the intensity of default is stochastic. We then give a calibration procedure that will be used to calibrate the drift term of the intensity of default to quotes of credit default swaps. This calibration can be done using closed form formulas if we assume that the short term interest rate r_t and the intensity of default λ_t are independent. If we assume that there is correlation between r_t and λ_t, the calibration can be carried out by solving an ordinary differential equation. We assume in this section that we have a quadratic Gaussian factor model for default free and defaultable securities[1].

We now consider a model where besides default free assets, defaultable assets are traded. We assume that we have a filtered probability space $(\tilde{\Omega}, \mathcal{G}, \mathbb{Q})$ where $\mathbb{Q}$ is the risk-neutral measure. We assume the filtration $\mathbb{F} = (\mathcal{F}_t)_{(0 \leqslant t \leqslant T*)}$ is generated by n independent Brownian motions $W(t) = W^i(t), i = 1, \ldots, n$ and satisfies the usual conditions. The initial filtration $\mathcal{F}_0$ is taken to be trivial. The time horizon is assumed to be finite so that $T^* > 0$ is some finite number. Let τ denote the default time of a corporate which

[1]See Assefa (2007) for the default free case.

has issued defaultable bonds. Let $H_t = \mathbb{1}_{\tau \leqslant t}$ represent the default indicator function. Let $\mathcal{H}_t = \sigma(H_u : u \leqslant t) = \sigma(\{\tau \leqslant u\} : u \leqslant t)$. We then define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. We denote by λ_t the $\mathcal{G}_t$-intensity of H_t which has the property that

$$H_t - \int_0^{t \wedge \tau} \lambda_u \, du \tag{1}$$

is a $\mathcal{G}$-martingale under the risk neutral measure $\mathbb{Q}$. Let the random vector

$$Y_t + \alpha(t) = (Y_{1t}, \dots, Y_{nt}) + (\alpha_1(t), \dots, \alpha_n(t))$$

follow a Gaussian Ornstein-Uhlenbeck process:

$$dY_t = AY_t \, dt + \Sigma \, dW_t \tag{2}$$

where $Y_0 = (0, \dots, 0)$ and A and Σ are constant square matrices. We assume that A is a diagonal matrix so that

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & \dots & a_{ii} & \dots & 0 \\ \vdots & 0 & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

and

$$\Sigma\Sigma^\top = \begin{bmatrix} \sigma_{11}^2 & \dots & \dots & \rho_{1n}\sigma_{11}\sigma_{1n} \\ \vdots & \ddots & \dots & \vdots \\ \rho_{i1}\sigma_{11}\sigma_{ii} & \dots & \dots \rho_{ij}\sigma_{ii}\sigma_{jj} & \dots \rho_{in}\sigma_{ii}\sigma_{nn} \\ \vdots & \dots & \ddots & \vdots \\ \rho_{1n}\sigma_{11}\sigma_{nn} & & \dots & \sigma_{nn}^2 \end{bmatrix}.$$

We now use quadratic forms in

$$Z_t = Y_t + \alpha(t) \tag{3}$$

Thus we assume that

$$r_t = Z_t^\top \widehat{C} Z_t + \widehat{B}^\top(t) Z_t + \widehat{A}(t) \tag{4}$$

and in the defaultable case we also have

$$\lambda_t = Z_t^\top \widetilde{C} Z_t + \widetilde{B}^\top(t) Z_t + \widetilde{A}(t) \tag{5}$$

where $\widehat{C}$ and $\widetilde{C}$ are constant and symmetric matrices, $\widehat{B}(t)$ and $\widetilde{B}(t)$ are time dependent deterministic vectors, $\widehat{A}(t)$ and $\widetilde{A}(t)$ are time dependent deterministic scalars. We denote by $D(t)$ the default free savings account which is the value of investing one unit of currency at time $t = 0$ and rolling over the account at the default free instantaneous rate of interest r_t:

$$D(t) = \exp\left(\int_0^t r_s \, ds\right). \tag{6}$$

Suppose G_T is some stochastic process and we are considering the expectation G_T with under a given measure $\mathbb{M}$ and with respect to the sigma algebra $\mathcal{F}_t$ representing the information from the market at time t given by

$$\mathbb{E}^{\mathbb{M}}[Q_T|\mathcal{F}_t].$$

To simplify the notation we write instead

$$\mathbb{E}_t^{\mathbb{M}}[Q_T].$$

We denote the price of a default free zero coupon bond by $P(t,T)$ which is given by(see, e.g., Musiela and Rutkowski (2005))

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s \, ds\right)\right].$$

We denote the predefault price of a defaultable zero coupon bond by $\bar{P}(t,T)$ so that under the assumption of the reduced form model we are considering, the price of a defaultable zero coupon bond is given by(see, e.g., Bielecki and Rutkowski (2002))

$$\mathbb{1}_{\tau>t}\bar{P}(t,T) = \mathbb{1}_{\tau>t}\mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s + \lambda_s \, ds\right)\right].$$

If we model the default free interest rate r_t and the intensity of default λ_t by quadratic forms of multivariate quadratic Gaussian processes, default free zero coupon bond prices $P(t,T)$ and defaultable zero coupon bond prices $\bar{P}(t,T)$ are log-quadratic Gaussian(see El Karoui et al. (1991) for example). Hence

$$P(t,T) = \exp\left(-Y_t^\top C(t,T) Y_t - B(t,T)^\top Y_t - A(t,T)\right) \quad (7)$$

$$\bar{P}(t,T) = \exp\left(-Y_t^\top \bar{C}(t,T) Y_t - \bar{B}(t,T)^\top Y_t - \bar{A}(t,T)\right) \quad (8)$$

where $C(t,T), \bar{C}(t,T)$ are symmetric matrices, $B(t,T), \bar{B}(t,T)$ are vectors and $A(t,T), \bar{A}(t,T)$ are scalars. In general if we consider the matrices A, Σ in (2) to be time dependent, we cannot guarantee the existence of a closed form[2]. However if we assume A, Σ are constant, we can give closed form formulas for $P(t,T)$ and $\bar{P}(t,T)$(see Assefa (2007) for the details). Let $V^d(t,T)$ denote the variance-covariance matrix of Y_T under the measure $\mathbb{T}^d$ conditional on $\mathcal{F}_t$ and $M^d(t,T)$ denote the vector representing the mean of Y_T under the measure $\mathbb{T}^d$ conditional on $\mathcal{F}_t$.

Lemma 2.1. *The following ODE's are satisfied by $V(t,T)$ and $M(t,T)$*

$$\partial_T V(t,T) = A(t)V(t,T) + V(t,T)\,A(t)^\top - 2V(t,T)\,\widehat{C}\,V(t,T) + \Sigma\Sigma^\top \quad (9)$$

$$\partial_T M(t,T) = A(t)M(t,T) - 2V(t,T)\,\widehat{C}\,M(t,T) - V(t,T)\,\widehat{B}(t) + \mu(T), \quad (10)$$

with initial conditions $V(t,t) = 0_{n\times n}$ and $M(t,t) = Y_t$.

Proof: See Cherif et al. (1994) for the proof of this lemma.

Let $\overline{M}(t,T)$ and $\bar{V}(t,T)$ denote the conditional mean and conditional variance of Y_t under $\bar{\mathbb{T}}$. We can solve for $\bar{V}(t,T)$ and $\overline{M}(t,T)$ explicitly by solving equations similar to the ones given in Lemma 2.1.

Definition 1. *A Credit Default Swap(CDS) is a security that guarantees the payment of a deterministic amount Z to the payer from the receiver at default time τ of a corporate if default occurs at or after $T_n \geqslant 0$ and before maturity $T = T_N > T_n$. In return the payer pays a constant premium K at specified dates $\mathcal{T} = T_{n+1}, \ldots, T_N$ if default has not occurred by time T_i. Let i be chosen such that T_i is the premium payment date immediately preceding $\tau \leqslant T$. Then if there is a default at time τ before the maturity T of the contract, then the contract is terminated after an accrued payment of $(\tau - T_i)K$ is made by the payer. We call the payment Z at default the protection payment.*

[2] By closed form we mean up to a one dimensional numerical integration for the values $B(t,T), \bar{B}(t,T)$ and $A(t,T), \bar{A}(t,T)$.

Let $\zeta(\tau) = \max[i : n+1 \leqslant i \leqslant N, T_i < \tau]$ and $\beta_i = T_i - T_{i-1}$. Then the price of a CDS at time $t \leqslant T_n$ is given by the following formula(see Bielecki and Rutkowski (2002))

$$CDS(t, \mathcal{T}, T, K, Z) = \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^\tau r_s \, ds \right) Z \, \mathbb{1}_{T_n < \tau \leqslant T} -$$

$$\exp\left(- \int_t^\tau r_s \, ds \right) (\tau - T_{\zeta(\tau)}) K \, \mathbb{1}_{T_n < \tau \leqslant T} - \sum_{i=n+1}^{N} \exp\left(- \int_t^{T_i} r_s \, ds \right) \beta_i \, K \, \mathbb{1}_{\tau > T_i} \Bigg| \mathcal{G}_t \Bigg]$$

Under the proper assumptions of a reduced model of default where the default time τ is the first jump time of a conditional Poisson process, we obtain

$$CDS(t, \mathcal{T}, T, K, Z) = \mathbb{1}_{\tau > t} Z \int_{T_n}^{T} \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^s r_k + \lambda_k \, dk \right) \lambda_s \Bigg| \mathcal{F}_t \Bigg] \, ds -$$

$$- \mathbb{1}_{\tau > t} K \int_{T_n}^{T} \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^s r_k + \lambda_k \, dk \right) \lambda_s \Bigg| \mathcal{F}_t \Bigg] (s - T_{\zeta(s)}) \, ds -$$

$$- \mathbb{1}_{\tau > t} K \sum_{i=n+1}^{N} \beta_i \, \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^{T_i} r_k + \lambda_k \, dk \right) \Bigg| \mathcal{F}_t \Bigg]$$

Using the defaultable bond $\mathbb{1}_{\tau > t} \bar{P}(t, s)$,

$$\mathbb{1}_{\tau > t} \bar{P}(t, s) = \mathbb{1}_{\tau > t} \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^s r_k + \lambda_k \, dk \right) \Bigg| \mathcal{F}_t \Bigg]$$

as the numeraire and denoting the corresponding defaultable forward measure by $\bar{\mathbb{T}}_s$ the price of a CDS is given by[3]

$$CDS(t, \mathcal{T}, T, K, Z) = \mathbb{1}_{\tau > t} Z \int_{T_n}^{T_N} \bar{P}(t, s) \, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_t] \, ds -$$

$$- \mathbb{1}_{\tau > t} K \int_{T_n}^{T_N} \bar{P}(t, s) \; \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_t] \; (s - T_{\zeta(s)}) \, ds - \mathbb{1}_{\tau > t} K \sum_{i=n+1}^{N} \beta_i \; \bar{P}(t, T_i). \quad (11)$$

[3]See Bielecki and Rutkowski (2002) for more detail.

As we can get $\bar{P}(t,T)$ in closed form, we only need to know how to calculate $\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_t]$. This can also be obtained in closed form under the quadratic Gaussian model using Lemma 2.2 given in Assefa (2007) as $\bar{P}(t,s)$ can be calculated using the results for the default free case.

We now show how we can calibrate the drift term of λ_t using credit default swap quotes for the corporation whose default time is denoted by τ. Under independence of the interest rate and the intensity of default, the probability of default is given by

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^s \lambda_k\, dk\right)\middle|\mathcal{F}_t\right]. \tag{12}$$

In order to calibrate λ_t, we need the term structure of default probabilities. Let c and d represent positive integers such that $\{i_1,\ldots,i_c\}$ and $\{i_1,\ldots,i_d\}$ represent disjoint subsets of $\{1,\ldots,n\}$ where here n refers to the dimension of Y_t. Now let

$$Z_t^1 = \alpha^1(t) + Y_t^1 = (\alpha_{i_1}(t),\ldots,\alpha_{i_c}(t)) + (Y_{i_1 t},\ldots,Y_{i_c t})$$

and

$$Z_t^2 = \alpha^2(t) + Y_t^2 = (\alpha_{j_1}(t),\ldots,\alpha_{j_d}(t)) + (Y_{j_1 t},\ldots,Y_{j_d t})$$

be used to model r_t and λ_t respectively where we assume that the instantaneous correlation matrix of Y_t^1 and Y_t^2 is a diagonal matrix. Therefore r_t and λ_t are assumed to have zero correlation and this means Y_t^1 and Y_t^2 are independent. If there is a liquid market for defaultable zero coupon bonds for a range of maturities as in the case of a default free bond, then we can extract the default probabilities and calibrate the drift term of Z_t^2 through $\alpha^2(t)$. In general the market for defaultable bonds issued by a corporate is not liquid. The market for credit default swaps where a corporate is the reference name underlying the credit default swap contract has better liquidity. However the maturities of credit default swaps traded are usually one year, three years, five years and ten years. Practitioners assume a piecewise constant intensity and use a bootstrapping procedure to extract the term structure of default probabilities. We can use these term structure of default probabilities to calibrate our model but this is in contradiction to the assumed stochastic intensity of default. Moreover this bootstrapping procedure has some disadvantages such as not being robust to unreliable quotes for some maturities. A better method for extracting the term structure of default probabilities is given in Martin et al. (2001). We give a brief description of this method. The method described in Martin et al. (2001) is for a time dependent deterministic λ_t. Here we extend this method to the case of a stochastic intensity of default λ_t under the assumption of independence between the default free short rate of interest and the intensity of default. We later give how we can still modify this method to the case where the short rate of interest and the intensity of default are not independent. Ignoring the accrued premium, the value of a credit default swap of maturity T to the seller at time $t = 0$ is given by

$$K \sum_{i=0}^{N} \beta_i \, \bar{P}(0, T_i) - Z \int_0^{T_N} \bar{P}(0, s) \, \mathbb{E}^{\mathbb{T}_s}[\lambda_s | \mathcal{F}_t] \, ds$$

where $0 = T_0, \ldots, T_N = T$ are the premium payment dates. The quoted CDS rates are chosen so that the value of the CDS is equal to zero at time $t = 0$. Therefore the premium K is chosen to be

$$R_f(T) := \frac{Z \int_0^{T_N} \bar{P}(0, s) \, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_t] \, ds}{\sum_{i=0}^{N} \beta_i \, \bar{P}(0, T_i)} \tag{13}$$

where we assume a notional of one unit of currency and a constant recovery rate δ such that $Z = 1 - \delta$. Assuming independence between r_t and λ_t, we can write $R_f(T)$ as

$$\begin{aligned} R_f(T) &= \frac{Z \int_0^{T_N} \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_0^s r_k + \lambda_k \, dk \Big) \lambda_s \Big| \mathcal{F}_t \Big] \, ds}{\sum_{i=0}^{N} \beta_i \, \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_0^{T_i} r_k + \lambda_k \, dk \Big) \Big| \mathcal{F}_t \Big]} \\ &= \frac{Z \int_0^{T_N} P(0, s) \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_0^s \lambda_k \, dk \Big) \lambda_s \Big| \mathcal{F}_t \Big] \, ds}{\sum_{i=0}^{N} \beta_i \, P(0, T_i) \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(- \int_0^{T_i} \lambda_k \, dk \Big) \Big| \mathcal{F}_t \Big]}. \end{aligned} \tag{14}$$

As described in Martin et al. (2001), we can discretize the interval $[0, T_N]$ by choosing $\Delta > 0$ and considering a set of dates $(0 = T_0, \ldots, T_j, \ldots, T_M =$

$T)$, $\Delta = T_j - T_{j-1}$, j=1,...,M. We can now approximate $R_f(T)$ by

$$R_f(T) = \frac{Z \sum\limits_{j=0}^{M} \frac{1}{2}(P(0,T_j) + P(0,T_{j+1}))\mathbb{E}^{\mathbb{Q}}\Big[\int\limits_{T_j}^{T_{j+1}} \exp -\Big(\int\limits_0^s \lambda_u \, du\Big)\lambda_s \, ds\Big]}{\sum\limits_{i=0}^{N} \beta_i \; P(0,T_i)\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int\limits_0^{T_i} \lambda_k \, dk\Big)\Big|\mathcal{F}_t\Big]}. \tag{15}$$

Let $G(0,t)$ denote the risk neutral probability of survival i.e. the probability there is no default between time zero and time t as seen with respect to the trivial filtration $\mathcal{F}_0$. Under the reduced form model we are considering, we can express $G(0,T)$ by the following formula

$$G(0,t) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int\limits_0^t \lambda_s \, ds\Big)\Big]. \tag{16}$$

Let the conditional default probability of default over (T_j, T_{j+1}) be denoted by g_j which is given by

$$g_j := \frac{\mathbb{E}^{\mathbb{Q}}[\tau > T_j] - \mathbb{E}^{\mathbb{Q}}[\tau > T_{j+1}]}{\mathbb{E}^{\mathbb{Q}}[\tau > T_j]} = 1 - \frac{G(0,T_{j+1})}{G(0,T_j)}. \tag{17}$$

Then we have

$$G(0,T_{j+1}) = G(0,T_j) - G(0,T_j)g_j, \quad G(0,0) = 1 \tag{18}$$

or as originally formulated in Martin et al. (2001) in terms of the risk neutral probability of default

$$H(0,T) := \mathbb{E}^{\mathbb{Q}}[\tau < T] = 1 - \mathbb{E}^{\mathbb{Q}}\Big[\exp -\Big(\int\limits_0^T \lambda_s \, ds\Big)\Big] \tag{19}$$

we have

$$H(0,T_{j+1}) = H(0,T_j) + (1 - H(0,T_j))g_j, \quad H(0,0) = 0. \tag{20}$$

Therefore the probability of default between T_j and T_{j+1} can be expressed in terms of $G(0,T_j)$ and g_j

$$\mathbb{E}^{\mathbb{Q}}\Big[\int\limits_{T_j}^{T_{j+1}} \lambda_s \exp\Big(-\int\limits_0^s \lambda_u \, du\Big)\Big] = G(0,T_j) - G(0,T_{j+1}) = G(0,T_j)g_j. \tag{21}$$

Therefore

$$R_f(T_N) = \frac{Z \sum_{0}^{M} \frac{1}{2}(P(0,T_j) + P(0,T_{j+1}))G(0,T_j)g_j}{\sum_{i=0}^{N} \beta_i \; \bar{P}(0,T_i)}. \qquad (22)$$

The ratio in the right hand side of (22) is a function of g_j such that for a given maturity T_k,

$$F(g_0,\dots,g_{k-1}) := \frac{Z \sum_{0}^{k-1} \frac{1}{2}(P(0,T_j) + P(0,T_{j+1}))G(0,T_j)g_j}{\sum_{i=0}^{k} \beta_i \; P(0,T_i)G(0,T_i)} \qquad (23)$$

If we assume that the CDS quotes $R_f(T_k)$, $k = 1,\dots,r$ are subject to a Gaussian error of σ_k, $k = 1,\dots,r$, the procedure suggested in Martin et al. (2001) is to minimize

$$W(g_0,\dots,g_{M-1}) = \nu \sum_{j=0}^{M-1} d(g_{j+1};g_j)^2 + \frac{1}{2}\sum_{k=1}^{r}\left(\frac{R_f(T_k) - F(g_0,\dots,g_{k-1})}{\sigma_k}\right)^2 \qquad (24)$$

where

$$d(g^{'};g) = \sqrt{(g^{'}-g)\log_e\left(\frac{g^{'}}{g}\right) + (g-g^{'})\log_e\left(\frac{1-g^{'}}{1-g}\right)} \qquad (25)$$

and ν is a positive constant which gives more smoothness for the default probability curve for higher values. The authors Martin et al. (2001) consider the case $\nu = 10, \nu = 10,000$ and $\sigma_k = 10^{-4}$, $k = 1,..r$ and time discretisations of $\Delta = 0.5$ and $\Delta = \frac{1}{6}$ corresponding to 6 months and 2 months. Thus we can extract the term structure of the probability of survival given by $G(0,T)$.

Now under the assumption of independence between r_t and λ_t, we have

$$\mathbb{E}^{\bar{\mathbb{T}}}[r_T] = \mathbb{E}^{\mathbb{Q}}\left[\frac{\exp\left(-\int_0^T r_s + \lambda_s\, ds\right)}{\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s + \lambda_s\, ds\right)\right]} r_T\right] \tag{26}$$

$$= \frac{\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s\, ds\right) r_T\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \lambda_s\, ds\right)\right]}{\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T r_s\, ds\right)\right]\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \lambda_s\, ds\right)\right]} \tag{27}$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{P(0,T)} \exp\left(-\int_0^T r_s\, ds\right) r_T\right] \tag{28}$$

$$= \mathbb{E}^{\mathbb{T}}[r_T]. \tag{29}$$

We calibrate the drift term of λ_t using

$$-\partial_T \log_e \bar{P}(0,T) = \mathbb{E}^{\bar{\mathbb{T}}}[r_T + \lambda_T]. \tag{30}$$

Under the assumption of independence equation (30) can be reduced to

$$-\partial_T \log_e P(0,T) - \partial_T \log_e G(0,T) = \mathbb{E}^{\mathbb{T}}[r_T] + \mathbb{E}^{\bar{\mathbb{T}}}[\lambda_T]. \tag{31}$$

The drift term of r_t is calibrated using a closed formula starting from the equation

$$-\partial_T \log_e P(0,T) = \mathbb{E}^{\mathbb{T}}[r_T]. \tag{32}$$

Therefore (31) simplifies to

$$-\partial_T \log_e G(0,T) = \mathbb{E}^{\bar{\mathbb{T}}}[\lambda_T]. \tag{33}$$

Under the assumption of independence the conditional mean and conditional variance(under $\bar{\mathbb{T}}$) of Y_t^1 and Y_t^2 which we denote by $\overline{M}^1(t,T), \bar{V}^1(t,T)$ and $\overline{M}^2(t,T), \bar{V}^2(t,T)$ respectively can be solved for separately. Note that under the assumption of independence $\overline{M}^1(t,T)$ is the same as the conditional mean of Y_t^1 under $\mathbb{T}$ and $\bar{V}^1(t,T)$ is the same as the conditional variance of Y_t^1 under $\mathbb{T}$. Here we only need to use the fact that we can solve $\overline{M}^1(t,T)$ independently from $\overline{M}^2(t,T)$. Thus if we assume for ease of exposition that

$$r_t = (Y_t^1 + \alpha^1(t))^\top (Y_t^1 + \alpha^1(t)) \tag{34}$$

$$\lambda_t = (Y_t^2 + \alpha^2(t))^\top (Y_t^2 + \alpha^2(t)), \tag{35}$$

we can proceed to calibrate the drift term of λ_t as in the proof of Theorem 3 of Assefa (2007).

Assuming correlation between the factors used to model r_t and the factors used to model λ_t, will require additional approximations to extract the term structure of probabilities of default under the forward measure $\mathbb{T}$ corresponding to using the default free bond of maturity T as the numeraire. Moreover each of the system of ODE's given in lemma 2.1 do not separate into two independent systems and we shall see that we have to resort to numerically solving a first order system of non-linear ODE's in the general case. First we assume that we have a discretisation as in the above so that the time interval T is divided into M subintervals of equal length Δ. Moreover if default occurs between $T_j < \tau \leqslant T_{j+1}$ payment of the constant amount Z is made at T_{j+1} for $j = 0, \ldots, M-1$. Under this assumption we can write(see Bielecki and Rutkowski (2002))

$$Z\,\mathbb{E}^{\mathbb{Q}}\Big[\exp-\Big(\int_0^{\tau} r_u\,du\Big)\mathbb{1}_{0<\tau\leqslant T}\Big] = Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u\,du\Big)\mathbb{1}_{T_j<\tau\leqslant T_{j+1}}\Big]$$

$$= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u\,du\Big)\mathbb{1}_{\tau>T_j}\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u\,du\Big)\mathbb{1}_{\tau>T_{j+1}}\Big]$$

$$= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_j} r_u+\lambda_u\,du\Big)\exp\Big(-\int_{T_j}^{T_{j+1}} r_u\,du\Big)\Big]-$$

$$-\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u+\lambda_u\,du\Big)\Big]$$

$$= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_j} r_u+\lambda_u\,du\Big)\exp\Big(-\int_{T_j}^{T_{j+1}} r_u\,du\Big)\Big]-$$

$$-\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u+\lambda_u\,du\Big)\Big]$$

$$= Z\sum_{j=0}^{M-1}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_j} r_u+\lambda_u\,du\Big)\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_{T_j}^{T_{j+1}} r_u\,du\Big)\Big|\mathcal{F}_{T_j}\Big]\Big]-$$

$$-\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_{j+1}} r_u+\lambda_u\,du\Big)\Big] \quad (36)$$

$$= Z\sum_{j=0}^{M-1}P(0,T_j)\mathbb{E}^{\mathbb{T}_j}\Big[\exp\Big(-\int_0^{T_j}\lambda_u\,du\Big)P(T_j,T_{j+1})\Big]-$$

$$-P(0,T_{j+1})\mathbb{E}^{\mathbb{T}_{j+1}}\Big[\exp\Big(-\int_0^{T_{j+1}}\lambda_u\,du\Big)\Big]. \quad (37)$$

The value $P(T_j, T_{j+1})$ is stochastic when seen from time $t = 0$, however if Δ is close to zero, $P(T_j, T_{j+1})$ is close to 1. Therefore it is close to being deterministic, and hence we would not introduce much error by assuming $\exp\Big(-\int_0^{T_{j+1}} \lambda_u\, du\Big)$ and $P(T_j, T_{j+1})$ are independent under the $\mathbb{T}_j$. We therefore approximate (37) by

$$Z \sum_{j=0}^{M-1} P(0,T_j)\mathbb{E}^{\mathbb{T}_j}\Big[\exp\Big(-\int_0^{T_j} \lambda_u\, du\Big)\Big]\mathbb{E}^{\mathbb{T}_j}\Big[P(T_j,T_{j+1})\Big] -$$

$$- P(0,T_{j+1})\mathbb{E}^{\mathbb{T}_{j+1}}\Big[\exp\Big(-\int_0^{T_{j+1}} \lambda_u\, du\Big)\Big]$$

$$= Z \sum_{j=0}^{M-1} P(0,T_{j+1})\Big(\mathbb{E}^{\mathbb{T}_j}\Big[\exp\Big(-\int_0^{T_j} \lambda_u\, du\Big)\Big] - \mathbb{E}^{\mathbb{T}_{j+1}}\Big[\exp\Big(-\int_0^{T_{j+1}} \lambda_u\, du\Big)\Big]\Big) \tag{38}$$

Let

$$\bar{G}(0,T) := \mathbb{E}^{\mathbb{T}}\Big[\exp\Big(-\int_0^{T} \lambda_u\, du\Big)\Big]$$

denote the probability of survival up to time T under the default free forward measure $\mathbb{T}$. Then

$$R_f(T) = \frac{Z \sum_{j=0}^{M-1} P(0,T_{j+1})(\bar{G}(0,T_j) - \bar{G}(T_{j+1}))}{\sum_{i=0}^{k} \beta_i\, P(0,T_i)\bar{G}(T_i)}. \tag{39}$$

Let the conditional default probability of default over (T_j, T_{j+1}) under the forward measure $\mathbb{T}$ be denoted by $\bar{g}_j$ which is given by

$$\bar{g}_j := \frac{\mathbb{E}^{\mathbb{T}_j}[\tau > T_j] - \mathbb{E}^{\mathbb{T}_{j+1}}[\tau > T_{j+1}]}{\mathbb{E}^{\mathbb{T}_j}[\tau > T_j]} = 1 - \frac{\bar{G}(0,T_{j+1})}{\bar{G}(0,T_j)}. \tag{40}$$

Then we have

$$\bar{G}(0,T_{j+1}) = \bar{G}(0,T_j) - \bar{G}(0,T_j)\bar{g}_j, \quad \bar{G}(0,0) = 1 \tag{41}$$

and $R_f(T)$ can be seen as a function of $(\bar{g}_0, \ldots, \bar{g}_M)$ and we can use an optimization procedure similar to the one that was used to extract $G(0,T)$ in case of independence between r_t and λ_t to extract $\bar{G}(T)$.

Once we have extracted $\bar{G}(T)$, we can obtain $\bar{P}(0,T)$ by using $\bar{P}(0,T) = P(0,T)\bar{G}(0,T)$. We can now use $\bar{P}(0,T)$ to calibrate the whole of $\alpha(t)$. Let us assume that r_t and λ_t are given by equations (34),(35) where we now assume that Y_t^1 and Y_t^2 are not independent. Then as in the case of the default free market(see Assefa (2007)) we have

$$\begin{aligned} -\partial_T \log_e \bar{P}(0,T) =& \mathbb{E}^{\bar{\mathbb{T}}}\Big[r_T + \lambda_T\Big] \\ =& tr(\bar{V}(0,T)) + (\bar{M}(0,T) + \alpha(T))^\top (\bar{M}(0,T) + \alpha(T)) \end{aligned} \tag{42}$$

where $\bar{M}(0,T)$ is the mean vector under the defaultable forward measure[4] $\bar{\mathbb{T}}$ of $Y_t = (Y_t^1, Y_t^2)$ used to model r_t and λ_t and $\bar{V}(0,T)$ is the variance-covariance matrix of Y_t under $\bar{\mathbb{T}}$. Let

$$\bar{F}(0,T) := -\partial_T \log_e \bar{P}(0,T) \tag{43}$$

denote the defaultable forward rate for maturity T at time $t = 0$ and

$$\widetilde{F}(T) := \sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T))}. \tag{44}$$

Hence (42) can be written as

$$\bar{F}(0,T) = tr(\bar{V}(0,T)) + (\overline{M}(0,T) + \alpha(T))^\top (\overline{M}(0,T) + \alpha(T)) \tag{45}$$

which is equivalent to

$$\widetilde{F}(T)^\top \widetilde{F}(T) = (\overline{M}(0,T) + \alpha(T))^\top (\overline{M}(0,T) + \alpha(T)). \tag{46}$$

To simplify the discussion we will now consider a two factor quadratic Gaussian model i.e. $Y_t = (Y_{1t}, Y_{2t})$ in (3). The first factor is used to model the short term rate of interest r_t so that we have $r_t = (Y_{1t} + \alpha_1(t))^2$. The second factor is used to model the intensity of default λ_t so that we have $\lambda_t = (Y_{2t} + \alpha_2(t))^2$. However the discussion that follows below can be easily extended to a multifactor quadratic Gaussian model where more than two factors are used for r_t or λ_t provided that the factors used to model r_t are

[4] this is measure that corresponds to using $\bar{P}(0,T)$ as the numeraire

not used to model λ_t and vice versa. We can first calibrate $\alpha_1(t)$ to the default free forward rate term structure at time $t = 0$ using a closed form formula(see Pelsser (2000) or Assefa (2007)). Next we need to calibrate to the defaultable forward rate term structure at time $t = 0$ through $\alpha_2(t)$. To continue our calibration procedure, we now have to consider how to linearize (46). If we were to define

$$\tilde{\tilde{F}}(T) = \begin{pmatrix} \tilde{\tilde{F}}_1(T) \\ \tilde{\tilde{F}}_2(T) \end{pmatrix} := \begin{pmatrix} \sqrt{\frac{\bar{F}(0,T) - Tr(\bar{V}(0,T))}{2}} \\ \sqrt{\frac{\bar{F}(0,T) - Tr(\bar{V}(0,T))}{2}} \end{pmatrix} \tag{47}$$

and proceed to derive the associated ODE for

$$\alpha(T) = \begin{pmatrix} \alpha_1(T) \\ \alpha_2(T) \end{pmatrix}$$

using the same techniques as in the proof of Theorem 3 in Assefa (2007), we get the following system of linear ODE's

$$\begin{aligned} \frac{d}{dT}\alpha_1(T) &= a_{11}\,\alpha_1(T) + a_{12}\,\alpha_2(T) - a_{11}\,\tilde{\tilde{F}}_1(0,T) - a_{12}\,\tilde{\tilde{F}}_2(0,T) \\ &\quad + 2\bar{V}_{11}(0,T)\tilde{\tilde{F}}_1(0,T) + 2\bar{V}_{12}(0,T)\tilde{\tilde{F}}_2(0,T) + \frac{d}{dT}\tilde{F}_1(0,T) \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{d}{dT}\alpha_2(T) &= a_{21}\,\alpha_1(T) + a_{22}\,\alpha_2(T) - a_{21}\,\tilde{\tilde{F}}_1(0,T) - a_{22}\,\tilde{\tilde{F}}_2(0,T) \\ &\quad + 2\bar{V}_{12}(0,T)\tilde{\tilde{F}}_1(0,T) + 2\bar{V}_{22}(0,T)\tilde{\tilde{F}}_2(0,T) + \frac{d}{dT}\tilde{F}_2(0,T) \end{aligned} \tag{49}$$

whose solution is given by

$$\alpha_1(T) = \tilde{\tilde{F}}_1(T) + 2\,\exp(a_{11}\,T)\int_0^T \exp(-a_{11}\,r)(\bar{V}_{11}(0,r)\,\tilde{\tilde{F}}_1(r) + \bar{V}_{12}(0,r)\,\tilde{\tilde{F}}_2(r))\,dr \tag{50}$$

$$\alpha_2(T) = \tilde{\tilde{F}}_1(T) + 2\,\exp(a_{22}\,T)\int_0^T \exp(-a_{22}\,r)(\bar{V}_{11}(0,r)\,\tilde{\tilde{F}}_1(r) + \bar{V}_{12}(0,r)\,\tilde{\tilde{F}}_2(r))\,dr. \tag{51}$$

However in order to calibrate to the default free term structure we have to use Theorem 3 in Assefa (2007)(or see Pelsser (2000) for the one factor case) which gives us

$$\alpha_1(T) = \tilde{F}(T) + 2\,\exp(a_{11}\,T)\int\limits_0^T \exp(-a_{11}\,r)V_{11}(0,r)\,\tilde{F}(r)\,dr \tag{52}$$

where $\tilde{F}(T) = \sqrt{F(0,T) - V(0,T)}$. The problem with defining $\tilde{\tilde{F}}(T)$ as in (47) is that the solution of (48) given in (50) does not guarantee that $\alpha_1(t)$ is equal to (52). Therefore we have to define $\tilde{\tilde{F}}$ in such a way that the $\alpha_1(t)$ obtained through the calibration procedure to the default free forward rate term structure does not change. In fact this leads to a unique way of defining a vector $\tilde{\tilde{F}}(T)$:

$$\tilde{\tilde{F}}(T) := \begin{pmatrix} \sqrt{H(T)} \\ \sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)} \end{pmatrix} \tag{53}$$

for some function $H(T)$ which we need to solve for in the following. Using the definition given in (53), we can linearize (46) and proceed to derive the associated ODE for $\alpha(t)$ by using procedure that was used in the proof of Theorem 3 in Assefa (2007). This gives us the ODE given in (48) whereby we now use the new definition (53) for $\tilde{\tilde{F}}(T)$. If $H(T)$ was a known function at this point, we can then proceed to solve (48) using the same steps used in the proof of Theorem 3 in Assefa (2007) to obtain

$$\begin{aligned}\alpha_1(T) =& \sqrt{H(T)} + 2\,\exp(a_{11}\,T)\int\limits_0^T \exp(-a_{11}\,r)(\bar{V}_{11}(0,r)\,\sqrt{H(r)} + \\ &+ \bar{V}_{12}(0,r)\sqrt{\bar{F}(0,r) - Tr(\bar{V}(0,r)) - H(r)}\;dr \end{aligned} \tag{54}$$

$$\begin{aligned}\alpha_2(T) =& \sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)} + \\ &+ 2\,\exp(a_{22}\,T)\int\limits_0^T \exp(-a_{22}\,r)(\bar{V}_{21}(0,r)\,\sqrt{H(r)} + \\ &+ \bar{V}_{22}(0,r)\sqrt{\bar{F}(0,r) - Tr(\bar{V}(0,r)) - H(r)}\;, dr. \end{aligned} \tag{55}$$

Since $H(T)$ is not known but $\alpha_1(t)$ is known, we can regard (54) as an integral equation for $H(T)$. Once we solve this equation we can give the solution for $\alpha_2(t)$ using (55). Since a solution of (54) is also a solution of the ODE given in (48) with $\tilde{\tilde{F}}(T)$ defined now as in (53), we will obtain an exact calibration to both the default free and defaultable term structures. The equation given in (54) is a nonlinear Volterra integral equation of the second kind in the unknown $H(T)$. We can convert (54) into a nonlinear first order differential equation using differentiation as the proof of the following theorem shows.

Theorem 1. *In a two factor quadratic Gaussian factor model, we have a perfect calibration to the default free term structure given by the price of default free zero coupon bonds through a closed form formula and a perfect calibration to the term structure of survival probabilities which are extracted from CDS quotes through the numerical solution of the following first order non-linear ODE:*

$$\frac{1}{2}\frac{d}{dT}H(T) + (2\,\bar{V}_{11}(0,T) - a_{11})H(T) + \Bigg((a_{11} - 2V(0,T))\widetilde{F}(T) - \frac{d}{dT}\widetilde{F}(T) + \\ + 2\,\bar{V}_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)}\Bigg)\sqrt{H(T)} = 0 \quad (56)$$

$H(T) > 0, \quad T \in [0, T^*], \quad H(0) = \alpha_1(0)^2.$

If we assume independence between r_t and λ_t, the exact solution of (56) *is given by*

$$H(T) = \tilde{F}(T) = \sqrt{F(0,T) - V(0,T)}$$

where $F(0,T)$ is default free instantaneous forward rate and $V(0,T)$ is the default free variance of Y_{1t} under $\mathbb{T}$.

Proof: As the discussion preceding this theorem shows that we only need to solve the Volterra integral equation of the second kind given in (54). We now show that (54) is equivalent to the solution of the first order non-linear ODE given by (56). From (54) we get the following equality

$$\alpha_1(T) - \sqrt{H(T)} = 2\,\exp(a_{11}\,T)\int_0^T \exp(-a_{11}\,r)(\bar{V}_{11}(0,r)\,\sqrt{H(r)} \\ + \bar{V}_{12}(0,r)\,\sqrt{\bar{F}(0,r) - Tr(\bar{V}(0,r)) - H(r)}\,dr. \quad (57)$$

We now derive both sides of the integral equation (54) to obtain

$$\frac{d}{dT}\alpha_1(T) = \frac{\frac{d}{dT}H(T)}{2\sqrt{H(T)}} + 2\,a_{11}\exp(a_{11}\,T)\int_0^T \exp(-a_{11}\,r)(\bar{V}_{11}(0,r)\sqrt{H(r)}$$
$$+\,\bar{V}_{12}(0,r)\sqrt{\bar{F}(0,r) - Tr(\bar{V}(0,r)) - H(r)})\,dr + 2\,V_{11}(0,T)\sqrt{H(T)}$$
$$+\,2\,V_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(V(0,T)) - H(T)}. \quad (58)$$

Using (57) to simplify (58), we get

$$\frac{\frac{d}{dT}H(T)}{2\sqrt{H(T)}} + a_{11}(\alpha_1(T) - \sqrt{H(T)}) + 2\,\bar{V}_{11}(0,T)\sqrt{H(T)}$$
$$+\,2\,\bar{V}_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)} - \frac{d}{dT}\alpha_1(T) = 0. \quad (59)$$

We can further simplify (59) by using

$$\frac{d}{dT}\alpha_1(T) = \frac{d}{dT}\widetilde{F}(T) + a_{11}(\alpha_1(T) - \widetilde{F}(T)) + 2\,V(0,T)\widetilde{F}(T) \quad (60)$$

which can be obtained from Theorem 3 in Assefa (2007) by derivation and using the fact that

$$\alpha_1(T) - \widetilde{F}(T) = 2\exp(a_{11}\,T)\int_0^T \exp(-a_{11}\,s)V(0,s)\widetilde{F}(s)\,ds. \quad (61)$$

This gives us

$$\frac{\frac{d}{dT}H(T)}{2\sqrt{H(T)}} - a_{11}\sqrt{H(T)} + 2\,\bar{V}_{11}(0,T)\sqrt{H(T)} +$$
$$+\,2\,\bar{V}_{12}(0,T)\sqrt{\bar{F}(0,T) - Tr(\bar{V}(0,T)) - H(T)} - \frac{d}{dT}\widetilde{F}(T) +$$
$$+\,a_{11}\widetilde{F}(T) - 2\,V(0,T)\widetilde{F}(T) = 0. \quad (62)$$

Assuming $H(T)$ is a positive function, we now multiply both sides of (62) by $\sqrt{H(T)}$ and rearrange the terms to get (56) given in the lemma. In general (56) can be solved efficiently using numerical methods for first order ODE's.

If we assume that the instantaneous correlation ρ between Y_{1t} and Y_{2t} is equal to zero i.e. the Brownian motions used to model Y_{1t} and Y_{2t} are independent, then

$$\bar{V}_{11}(0,T) = V(0,T) \tag{63}$$
$$\bar{V}_{12}(0,T) = 0. \tag{64}$$

is true. Therefore (56) becomes a simpler equation given by

$$\frac{1}{2}\frac{d}{dT}H(T) + (2\,\bar{V}_{11}(0,T) - a_{11})H(T) + \\ + \left((a_{11} - 2V(0,T))\widetilde{F}(T) - \frac{d}{dT}\widetilde{F}(T)\right)\sqrt{H(T)} = 0 \tag{65}$$

It is easy to verify that

$$H(T) = F(0,T) - V(0,T) = \tilde{F}(T)^2$$

is a solution of (65) by substituting this value into the ODE. Substituting $\tilde{F}(T)^2$ for $H(T)$ in (55), we obtain $\alpha_2(T)$. Note that in this case

$$\bar{F}(0,T) = F(0,T) + G(0,T)$$

and

$$\sqrt{\bar{F}(0,T) - V(0,T) - \bar{V}_{22}(0,T) - \tilde{F}(T)^2} = \sqrt{G(0,T) - \bar{V}_{22}(0,T)}.$$

Hence $\alpha_1(T)$ is calibrated to the default free forward rate term structure while $\alpha_2(T)$ is calibrated to the term structure of survival probabilities $G(0,T)$.

Remark 2. *Assuming that we use different factors to model r_t and λ_t, we can extend Theorem 1 to a quadratic Gaussian model where more than two factors are used to model r_t and λ_t. From the discussion preceding Theorem 1, we can see that we have to consider a vector $H(T)$ and using the method used in the proof of Theorem 1, we obtain a system of non-linear ODE's which have to be solved numerically.*

We now give numerical results to show that we can calibrate the drift terms of r_t and λ_t using CDS quotes and the default free term structure extracted from zero coupon bonds and default free swap rates. We still consider the two factor model where $r_t = (Y_{1t} + \alpha_1(t))^2$ and $\lambda_t = (Y_{2t} + \alpha_2(t))^2$ to carry out the numerical work. In the case of independence between r_t and λ_t we can use closed form formulas to do this calibration. Therefore we consider the case when r_t and λ_t are not independent where we have to numerically solve the nonlinear scalar ODE given by (56) in Theorem 1. We use the default free zero coupon bond data given in Table 1 and the CDS quotes data given in Table 2 which is obtained from Martin et al. (2001) to test the calibration. We assume that the recovery rate is 30% and therefore the default payment Z is equal to $1 - 0.3 = 0.7$. For example the first row of Table 2 states that the market CDS rate for a one year protection against default is 0.0045 which is equivalent to a quote of 45 basis points. Using the optimization procedure described in this section(see (36) and the following paragraphs), we can extract the probability of default under the default free forward measure which we denoted by $\bar{G}(0,T)$ by using just the default free zero coupon data and the CDS quotes. The probability of default under $\mathbb{T}$ is obtained by just assuming correlation between r_t and λ_t without having to specify a specific value. We assume that the premium payments are made annually and therefore $\beta_i = 1$, for $i = 1, \ldots, r$ when the CDS has a maturity of r years. Since requiring a smoother $\bar{G}(0,T)$ will make the optimization procedure favor smoothness instead of matching exactly the CDS quotes, we need to have a higher level of discretization of the integral corresponding to the default leg of the CDS(see (13) and the paragraph before (36)). Therefore we chose $\Delta = 0.0625$ which is smaller than the values used for Δ in Martin et al. (2001) where λ_t is assumed to be deterministic. The value of $\nu = 10$ was chosen for the smoothness parameter and $\sigma_k = 10^{-4}, k = 1, \ldots, r$ in the objective function to minimize which is given in equations (24) and (25). In the case of correlation between r_t and λ_t, we have to make more approximations to the CDS rate as demonstrated in (36). Therefore what we are calibrating to is not the exact CDS rate but an approximation. In Table 3 we give this approximation next to the exact CDS quotes under the column with heading "Calib CDS". In the last column of table 3, we give the value of the CDS rate in a two factor quadratic Gaussian model whereby we use the formula given in (13) to calculate this value. To use (13) we have to assume specific values for the speed of mean reversion matrix A, the instantaneous volatility Σ and the correlation ρ in the two factor model we are considering:

$$dZ_t = (\alpha(t) + AY_t)\, dt + \Sigma\, dW_t \tag{66}$$

T	$P(0,T)$
0.	1
1	0.93182
2	0.866762
3	0.806772
4	0.750876
5	0.699114
6	0.650255
7	0.604807
8	0.562855
9	0.523594
10	0.487314

Table 1: Zero Rates

where A and Σ are constant square matrices given by

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix},$$

$$\Sigma\Sigma^\top = \begin{bmatrix} \sigma_{11}^2 & \rho\sigma_{11}\sigma_{22} \\ \rho\sigma_{11}\sigma_{22} & \sigma_{22}^2 \end{bmatrix}.$$

In practice these parameters are obtained by calibrating to default free option data such as caps, floors or swaptions as well as options on credit sensitive securities such as options on credit default swaps. However this task is made difficult by the fact that options on credit sensitive securities is not liquidly available. Here we would like to show that for a high value of instantaneous correlation between Y_{1t} and Y_{2t}, we can still calibrate well to CDS quotes. We know that we have exact calibration to the default free term structure under the quadratic Gaussian model and hence this shows that we can also calibrate to the defaultable term structure even if r_t and λ_t are not independent. The specific parameters chosen are:

$$a_{11} = 0.01, a_{22} = 0.03, \sigma_{11} = 0.02, \sigma_{22} = 0.04, \rho = 0.9. \tag{67}$$

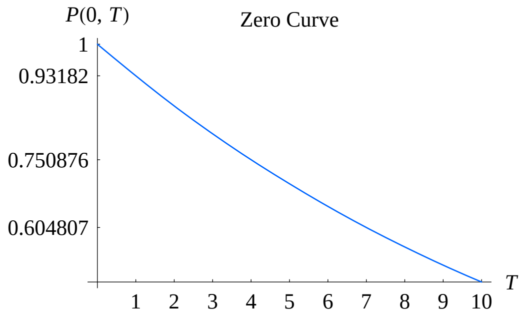

Figure 0.2.1: Discount curve

T	CDS rate
1	0.0045
2	0.0055
3	0.0065
4	0.0070
5	0.0095
7	0.0105
10	0.0115

Table 2: CDS quotes

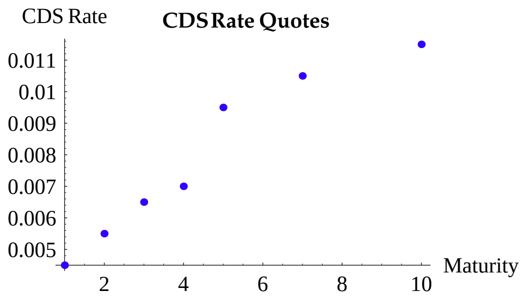

Figure 0.2.2: CDS quotes given as basis points $* 10^{-4}$

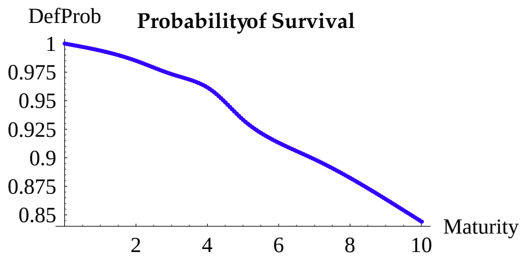

Figure 0.2.3: Extracted Survival Probability Under Correlation $\Delta = 0.0625$

Mat.	CDS Quote	Calib CDS	QG CDS
1	0.0045	0.00448017	0.00449686
2	0.0055	0.00547543	0.00551608
3	0.0065	0.00647092	0.00655311
4	0.007	0.00696875	0.00710304
5	0.0095	0.00945755	0.00967555
7	0.0105	0.010453	0.0108924
10	0.0115	0.0114484	0.0122455

Table 3: Calibration results to CDS quotes

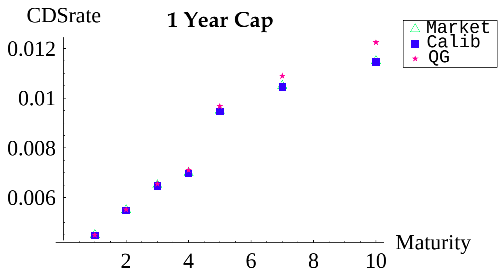

Figure 0.2.4: Calibration results to CDS quotes

Unlike the multifactor affine factor model for default free and defaultable markets where we do not have closed form formulas for calibration, the multi-factor quadratic Gaussian model for default free and defaultable markets enable us to calibrate exactly to the default free and defaultable term structures as the results in Table 3 and Figure 0.2.4 show. However for a numerical procedure of calibrating multifactor affine model to default free bonds and CDS quotes see Brigo and Alfonsi (2004).

3 Pricing Credit Default Swaptions

In this section we discuss the pricing of an option on a CDS also known as a credit default swaption by some members of the financial market. First we show how we can calculate the price of a credit default swaption accurately in a multivariate quadratic Gaussian factor model. We then derive different analytic approximations for the price of a credit default swaptions. In order to get closed form formulas, we will assume as in previous sections that the dynamics of the Gaussian Ornstein Uhlenbeck process Y_t is given by

$$dY_t = AY_t\, dt + \Sigma\, dW_t \tag{68}$$

where A is a constant diagonal matrix and Σ is a constant matrix and the state variables are given by $Z_t = \alpha(t) + Y_t$.

Definition 2. *An option to enter a credit default swap at a future time T_n gives the buyer of the option the right but not the obligation to enter into a CDS agreement with the receiver at time T_n by paying a premium of K at times $\mathcal{T}_{n,N} = T_{n+1}, \ldots, T_N$ in return for a protection payment of Z if the referenced credit defaults before the maturity $T_N > T_n$ of the credit default swap. At default time $T_n < \tau \leqslant T_N$ the contract is terminated and the receiver receives the accrued amount $\tau - T_i$ where T_i is the payment immediately preceding τ. This option contract is only valid if the reference credit does not default until time T_n. If the reference credit defaults by time T_n, the option contract is terminated with no exchange of payments.*

Only if at the time of maturity of the T_n the prevailing market CDS rate $R_f(T_n)$ is above K will the buyer of the option find it beneficial to exercise this option. The market CDS rate is set in such a way such that

$$CDS(T_n, \mathcal{T}_{n,N}, T, R_f(T_n), Z) = 0.$$

Hence at time T_n, the value of the CDS underlying the option which is given by

$$CDS(T_n, \mathcal{T}_{n,N}, T, K, Z) \tag{69}$$

will be positive if $K < R_f(T_n)$. Therefore the payoff of the option at time T_n

$$CDS(T_n, \mathcal{T}_{n,N}, T, K, Z) \tag{70}$$

will need to be positive so that the buyer of the credit default swaption finds it beneficial to exercise the option. Let the value of the credit default swaption be denoted by $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$, then at time $t < T_n$

$$\begin{aligned} CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) := \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^{T_n} r_s \, ds \right) \times \\ \times \left(CDS(T_n, \mathcal{T}_{n,N}, T, K, Z)\right)^{+} | \mathcal{F}_t \Bigg] \end{aligned} \tag{71}$$

Ignoring the accrued premium term this becomes

$$\begin{aligned} CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^{T_n} r_s \, ds \right) \times \\ \times \mathbb{1}_{\tau > T_n} \left(Z \int_{T_n}^{T_N} \bar{P}(T_n, s) \, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_{T_n}] \, ds - K \sum_{i=n+1}^{N} \beta_i \, \bar{P}(T_n, T_i) \right)^{+} \Bigg| \mathcal{F}_t \Bigg] \\ = \mathbb{1}_{\tau > t} \mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(- \int_t^{T_n} r_s + \lambda_s \, ds \right) \left(Z \int_{T_n}^{T_N} \bar{P}(T_n, s) \, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_{T_n}] \, ds - \\ - K \sum_{i=n+1}^{N} \beta_i \, \bar{P}(T_n, T_i) \right)^{+} \Bigg| \mathcal{F}_t \Bigg] = \mathbb{1}_{\tau > t} \bar{P}(t, T_n) \mathbb{E}^{\bar{\mathbb{T}}_n} \Bigg[\Bigg(Z \int_{T_n}^{T_N} \bar{P}(T_n, s) \, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_{T_n}] \, ds - \\ - K \sum_{i=n+1}^{N} \beta_i \, \bar{P}(T_n, T_i) \Bigg)^{+} \Bigg| \mathcal{F}_t \Bigg] \end{aligned} \tag{72}$$

where $\bar{\mathbb{T}}_n$ is used to denote $\bar{\mathbb{T}}_{T_n}$.

The value of the CDS underlying the credit default swaption in (72) is given by

$$Z \int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s | \mathcal{F}_{T_N}]\, ds - K \sum_{i=n+1}^{T_N} \beta_i\, \bar{P}(T_n, T_i). \tag{73}$$

The value (73) is random as seen with respect to the filtration $\mathcal{F}_t$ since Y_{T_n} is part of the formulas for $\bar{P}(T_n, s), \bar{P}(T_n, T_i)$ and $\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]$. However given Y_{T_n}, we can calculate (73) to a great degree of accuracy in the quadratic Gaussian factor model. Since Y_{T_n} is a Gaussian Ornstein Uhlenbeck process and we know the mean and variance-covariance matrix of Y_{T_n} under the defaultable forward measure $\bar{\mathbb{T}}_n$(see Lemma 2.1), we can use Monte Carlo simulation to calculate $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$. In Brigo & Alfonsi (2004), it is indicated that we need a large number of Monte Carlo simulations because a CDS has a large variance. The conditional distribution of Y_{T_n} under $\bar{\mathbb{T}}_n$ and conditional on $\mathcal{F}_t$ is multivariate Gaussian and we know the mean and variance in closed form. Therefore if the dimension of the Y_{T_n} is of low order, we can calculate $CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)$ much faster using multidimensional integration by directly integrating the payoff (73) times the multivariate normal distribution representing the probability density function of Y_{T_n} under $\bar{\mathbb{T}}_n$:

$$\begin{aligned} &CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \\ &\bar{P}(t, T_n) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(Z \int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds - K \sum_{i=n+1}^{T_N} \beta_i\, \bar{P}(T_n, T_i) \right)^+ \times \\ &\times \frac{1}{2\pi |\bar{V}(0, T_\alpha)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2} (Y_{T_\alpha} - \overline{M}(0, T_\alpha))^\top \bar{V}(0, T_\alpha)^{-1} (Y_{T_\alpha} - \overline{M}(0, T_\alpha)) \right) dY_{T_\alpha} \end{aligned} \tag{74}$$

where we have to discretize the integral

$$\int_{T_n}^{T_N} \bar{P}(T_n, s)\, \mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\, ds. \tag{75}$$

The multidimensional integral (74) can be done efficiently by using cubature techniques as shown in Cools and Haegemans (2003).

We have indicated how we can calculate the price of a credit default swaption to a great degree of accuracy if we use multidimensional integration or Monte Carlo simulation. However this task can be computationally demanding for a multifactor model where the number of factors used to model the interest rate and intensity is more than two. Moreover even for a low factor model it is be better to find a faster way of calculating the price of a credit default swaption compared to using cubature or Monte Carlo simulation. We now show how to approximate the price of a credit default swaption using closed formulas involving the numerical inversion of Fourier transforms. To derive the first analytic approximation, we first rewrite (72) as

$$CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) =$$

$$\mathbb{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\left[\sum_{i=n+1}^{N}\beta_i\;\bar{P}(T_n,T_i)\left(\frac{Z\int_{T_n}^{T_N}\bar{P}(T_n,s)\;\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\,ds}{\sum_{i=n+1}^{T_N}\beta_i\;\bar{P}(T_n,T_i)}-K\right)^+\middle|\mathcal{F}_t\right]. \tag{76}$$

We now approximate the integral

$$\int_{T_n}^{T_N}\bar{P}(T_n,s)\;\mathbb{E}^{\bar{\mathbb{T}}_s}[\lambda_s|\mathcal{F}_{T_n}]\,ds \tag{77}$$

using a right Riemann sum. Hence we divide the interval $[T_n, T_N]$ into M subintervals by choosing $\delta = \frac{T_N - T_n}{M}$ and $T_{n+j} = T_n + \delta j$, $j = 1, \dots, M$ to get

$$\frac{\delta Z\sum_{j=n+1}^{N}\bar{P}(T_n,T_j)\;\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]\,ds}{\sum_{i=n+1}^{T_N}\beta_i\;\bar{P}(T_n,T_i)} \tag{78}$$

where $\bar{\mathbb{T}}_j$ is used to denote $\bar{\mathbb{T}}_{T_j}$. Using

$$\bar{w}_j(T_n) := \frac{\bar{P}(T_n,T_j)}{\sum_{i=n+1}^{T_N}\beta_i\;\bar{P}(T_n,T_i)}, j = n+1, \dots, N \tag{79}$$

and ignoring the error introduced by the discretization (78), we can now express (76) as

$$
\begin{aligned}
&CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \\
&\mathbb{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\Bigg[\sum_{i=n+1}^{N}\beta_i\ \bar{P}(T_n,T_i)\left(\frac{\delta Z\sum\limits_{j=n+1}^{N}\bar{w}_j(T_n)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]}{\sum\limits_{i=n+1}^{N}\beta_i\ \bar{P}(T_n,T_i)} - K\right)^+\Bigg|\mathcal{F}_t\Bigg] \\
&= \mathbb{1}_{\tau>t}\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(-\int_t^{T_n} r_s+\lambda_s\,ds\right)\sum_{i=n+1}^{N}\beta_i\ \bar{P}(T_n,T_i)\times \\
&\qquad\times\left(\frac{\delta Z\sum\limits_{j=n+1}^{N}\bar{w}_j(T_n)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]}{\sum\limits_{i=n+1}^{N}\beta_i\ \bar{P}(T_n,T_i)} - K\right)^+\Bigg|\mathcal{F}_t\Bigg]. \qquad (80)
\end{aligned}
$$

In the following we consider a change of measure that would simplify the calculation of (80). First we define the defaultable present value of a basis point(DPVBP) by

$$
U_{n,N}(t) := \mathbb{1}_{\tau>T_n}\sum_{i=n+1}^{N}\beta_i\bar{P}(t,T_i). \qquad (81)
$$

Consider the measure that is absolutely continuous to to $\mathbb{Q}$ which is defined by the Radon-Nikodým density:

$$
\left.\frac{d\mathbb{U}}{d\mathbb{Q}}\right|_{\mathcal{G}_{T_n}} = \frac{U_{n,N}(T_n)}{D(T_n)}\frac{D(0)}{U_{n,N}(0)} = \mathbb{1}_{\tau>T_n}\frac{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{D(T_n)}\frac{D(0)}{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(0,T_i)}. \qquad (82)
$$

Now using the abstract Bayes formula and (82) we can rewrite (80) as

$$
\begin{aligned}
&CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \\
&\qquad U_{n,N}(t)\,\mathbb{E}^{\mathbb{U}}\Bigg[\left(\delta Z\sum_{j=n+1}^{N}\bar{w}_j(T_n)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] - K\right)^+\Bigg|\mathcal{F}_t\Bigg]. \qquad (83)
\end{aligned}
$$

We now show how we can calculate

$$\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]. \tag{84}$$

Under the assumption of a multifactor quadratic Gaussian model, the interest rate and the intensity of default are quadratic forms in Y_T. We partition the vector Y_T into two disjoint parts consisting of Y_T^1 and Y_T^2 and for ease of exposition assume r_T and λ_T are given by (34) and (35) respectively. We denote by $\overline{M}^i(t,T)$ and $\bar{V}^i(t,T), i = 1,2$ the conditional mean and variance-covariance matrix of $Y_T^i, i = 1,2$ under the measure $\bar{\mathbb{T}}_T$ and with respect to the sigma field $\mathcal{F}_t$. Using Lemma 2.1, we can find the mean and variance-covariance matrix of Y_T under the measure $\bar{\mathbb{T}}_T$ and conditional on $\mathcal{F}_t$ which are denoted by $\overline{M}(t,T)$ and $\bar{V}(t,T)$ respectively:

$$\bar{V}(t,T) = \overline{Pv}(T-t)\overline{Qv}^{-1}(T-t) \tag{85}$$

$$\overline{M}(t,T) = \overline{Qv}^{-1}(t,T)^\top Y_t + 2\overline{Qv}^{-1}(t,T)^\top \int_t^T \overline{Pv}(t,s)^\top \alpha(s)\, ds \tag{86}$$

where $(\overline{Qv}(T), \overline{Pv}(T))^\top$ is the solution of the following system of linear differential equations

$$\begin{pmatrix} \partial_T \overline{Qv}(T) \\ \partial_T \overline{Pv}(T) \end{pmatrix} = \begin{pmatrix} -A^\top & 2I \\ \Sigma\Sigma^\top & A \end{pmatrix} \begin{pmatrix} \overline{Qv}(T) \\ \overline{Pv}(T) \end{pmatrix}.$$

So if in particular $t = T_n$ and $T = T_j$, we can find the mean and variance-covariance matrix of $Y_{T_j}^2$ under the measure $\bar{\mathbb{T}}_j, T_j > T_n$ and with respect to the sigma field $\mathcal{F}_{T_n}$ which we denoted by $\overline{M}^2(T_n,T_j)$ and $\bar{V}^2(T_n,T_j)$ from $\overline{M}(T_n,T_j)$ and $\bar{V}(T_n,T_j)$. This shows that we can find the value of (84) in closed form. The mean and variance-covariance matrix of Y_{T_n} under $\mathbb{U}$ can be calculated explicitly. Using (82) and the abstract Bayes formula we have

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{U}}[Y_{T_j}] =& \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-\int_t^{T_n} r_s+\lambda_s\,ds\right)\frac{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(t,T_i)}Y_{T_j}\right] \qquad (87)\\
=& \mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\sum\limits_{i=n+1}^{N}\beta_i\bar{P}(T_n,T_i)}{\sum\limits_{i=n+1}^{N}\beta_i\frac{\bar{P}(t,T_i)}{\bar{P}(t,T_n)}}Y_{T_j}\right]\\
=& \sum_{i=n+1}^{N}\frac{\beta_i\bar{P}(t,T_i)}{\sum_{i=n+1}^{N}\beta_i\bar{P}(t,T_i)}\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\frac{\bar{P}(t,T_i)}{\bar{P}(t,T_n)}}Y_{T_j}\right]\\
=& \sum_{i=n+1}^{N}\frac{\beta_i\bar{P}(t,T_i)}{\sum_{i=n+1}^{N}\beta_i\bar{P}(t,T_i)}\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}Y_{T_j}\right]. \qquad (88)
\end{aligned}
$$

We can calculate

$$
\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}Y_{T_j}\right] \qquad (89)
$$

by using the result of Cherif et al. (1994)(or see lemma 5.3 in Assefa (2007)) as $\bar{P}(T_n,T_i)$ is log-quadratic Gaussian. Therefore (87) can be calculated in closed form. Note however that we can use Girsanov's theorem to show that the dynamics of Y_t does not correspond to a Gaussian process under the measure $\mathbb{U}$ even though we can get the mean and variance-covariance matrix as the weighted mean and variance-covariance matrix of a Gaussian Y_t under each measure $\bar{\mathbb{T}}_i$ as:

$$
\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n,T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n,T_i)]}Y_{T_j}\right] = \mathbb{E}_t^{\bar{\mathbb{T}}_i}[Y_{T_j}].
$$

To facilitate the derivation of an analytic approximation to (83), we now replace the weights $\bar{w}_j(T_n)$ by their time zero values. This freezing of the weights has been used to derive analytic approximations in the valuation of default free securities(see **?**)) and in the valuation of defaultable securities(see Brigo and Mercurio (2006)). Therefore

$$
\bar{Q}_\lambda(T_n) := \delta Z\sum_{j=n+1}^{N}\bar{w}_j(0)\bar{Q}_{\lambda_i} = \delta Z\sum_{j=n+1}^{N}\bar{w}_j(0)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \qquad (90)
$$

is a quadratic form in $Y_{T_n} = (Y^1_{T_n}, Y^2_{T_n})$. We now made an additional approximation by replacing Y_t under $\mathbb{U}$ by a Gaussian process $\widetilde{Y}_t$ which has mean and variance-covariance matrix equal to the exact mean and variance-covariance matrix of Y_t under $\mathbb{U}$. Therefore an approximation to the credit default swaption can be given by

$$CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \approx \widetilde{CDS}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = U_{n,N}(t)\mathbb{E}^{\mathbb{U}}\left[\left(\delta Z \sum_{j=n+1}^{N} \bar{w}_j(0)\, Q_F(\widetilde{Y}_{T_n}) - K\right)^+ \middle| \mathcal{F}_t\right] \quad (91)$$

where the $Q_F(\widetilde{Y}_{T_n})$ denotes the quadratic form in $\widetilde{Y}_{T_n}$ which is obtained from (91) by replacing Y_{T_n} in (90) by $\widetilde{Y}_{T_n}$. This approximation is an efficient one since we only have to work with the characteristic function of a single quadratic form in Gaussian random variables[5]. Under this assumption (90) can be approximated by a quadratic form in Gaussian random variables. Since we can calculate the characteristic function of quadratic forms in Gaussian random variables we can now apply the Fourier transform method which is based on transforming the approximate price of the credit default swaption which is given by (91) with respect to the strike price(see Carr and Madan (1998),Lee (2004)). This is similar to what we have done in approximating the price of swaptions in Assefa (2007) but here we do not have to approximate a sum of log-quadratic Gaussian processes by a log-quadratic process but only freeze the weights $\bar{w}_j(t)$ and approximate the non Gaussian dynamics of Y_t under $\mathbb{U}$ by a Gaussian process with matching mean and variance-covariance matrix. For ease of exposition we now discuss how to calculate (91) for $t = 0$. Let $\mathfrak{Q}$ denote a quadratic form in Gaussian random variables and $\Phi^{\mathbb{U}}(\mathfrak{Q}, z)$ denote the characteristic function of $\mathfrak{Q}$ under $\mathbb{U}$

$$\Phi^{\mathbb{U}}(\mathfrak{Q}, z) := \mathbb{E}^{\mathbb{U}}[\exp(iz\mathfrak{Q})]. \quad (92)$$

The payoff function associated with (91) is of type

$$G_2(x, k) := (x - k) + . \quad (93)$$

[5]When discussing the numerical results later in this section, we will show that we can find the exact characteristic function of (90) under $\mathbb{U}$ but it turns out in addition to be computationally less efficient, the approximation tends to have generally more error.

For such a payoff function Lee (2004) gives a method to calculate the inverse transform with error bounds. Let $\hat{\alpha} > 0$ and $\mathcal{C}_{\hat{\alpha},G_2}(K)$ denote the dampened price of the option[6] with payoff $G_2(x,k)$

$$\mathcal{C}_{\hat{\alpha},G_2}(K) := \exp(\hat{\alpha}K)\mathbb{E}^{\mathbb{U}}[G_2(x,K)]. \tag{94}$$

Let $\widehat{\mathcal{C}}_{G_2}(z)$ denote the Fourier transform of the damped option price with respect to the strike price K

$$\hat{\mathcal{C}}_{G_2}(z) := \int_{-\infty}^{\infty} \exp(izK)\,\mathcal{C}_{\hat{\alpha},G_2}(K)\;dK = \frac{-\Phi^U(x,z)}{z^2}. \tag{95}$$

Then from Lee (2004) the option price can be obtained by the following Fourier inversion

$$\mathcal{C}_{G_2}(K) = R_{\hat{\alpha},G_2} + \frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}\Big[\hat{\mathcal{C}}_{G_2}(z)\exp(-izK)\Big]\,dz$$

where

$$R_{\hat{\alpha},G_2} = \begin{cases} -\bar{\Phi}'(0) - K\bar{\Phi}(0), & \hat{\alpha} < 0 \\ \frac{-\bar{\Phi}'(0)-K\bar{\Phi}(0)}{2}, & \hat{\alpha} = 0 \\ 0, & \hat{\alpha} > 0 \end{cases} \tag{96}$$

If $\hat{\alpha} = 0$, Lee (2004) suggests that we use

$$\mathcal{C}_{G_2}(K) = R_{\hat{\alpha},G_2} + \frac{1}{\pi}\int_{0}^{\infty} \mathrm{Re}\Big[\hat{\mathcal{C}}_{G_2}(z)\exp(-izK)\Big] + \frac{1}{z^2}\,dz$$

to avoid convergence problems[7]. The characteristic function of a quadratic Gaussian variable exists everywhere and therefore we can choose to dampen the option price or not. This enables us a choice of methods to minimize the error in the numerical inversion of the Fourier transform. For further details see Lee (2004). Thus we can calculate (91) in closed form requiring only the numerical inversion of a Fourier transform.

[6] We mean here the price of the credit default swaption divided by the predefault value of the defaultable present value of a basis point.

[7] Numerical experiments show that there is still some difficulty when using $\hat{\alpha} = 0$.

We now derive another approximation which uses the quadratic form given in (90) to approximate the exercise boundary of the credit default swaption. From the first approximation to the price of a credit default swaption which is given by (91), we can see that the credit default swaption is exercised if

$$\bar{Q}_{n,N}(T_n) := \delta Z \sum_{j=n+1}^{N} \bar{w}_j(0)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \tag{97}$$

is greater than the exercise price K. We have already seen in the derivation of the first approximation that $\bar{Q}_{n,N}(T_n)$ is a quadratic form in T_n which can be calculated in closed form(see the discussion after (91)). Ignoring the accrued term, the exact price of the credit default swaption is given by (71). If we now discretize the integral corresponding to the default leg using a Riemann sum as in (78), we can approximate[8] the exact price of the credit default swaption(see (71)) under $\bar{\mathbb{T}}_n$ by

$$\begin{aligned} &CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \approx \\ &\quad \mathbb{1}_{\tau>t}\bar{P}(t,T_n)\mathbb{E}^{\bar{\mathbb{T}}_n}\Bigg[\Bigg(\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n,T_j)\ \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] - \\ &\qquad\qquad - K \sum_{i=n+1}^{N} \beta_i\ \bar{P}(T_n,T_i)\Bigg)^+ \Bigg| \mathcal{F}_t\Bigg]. \end{aligned} \tag{98}$$

As (98) is close to the exact price of the credit default swaption, we can approximate the exact price of a credit default swaption with maturity $T_n = T$ by approximating the exercise boundary through (97). Therefore we get a second approximation to the price of the credit default swaption which is

[8]We can assume that this error can be ignored as we can approximate the integral with high degree of accuracy for a value of the discretization level δ which is close enough to zero.

given by:

$$
\begin{aligned}
&CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \approx \widetilde{CDS2}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) \\
&\quad = \mathbb{1}_{\tau>t} \bar{P}(t, T_n) \mathbb{E}^{\bar{\mathbb{T}}_n}\Bigg[\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n, T_j)\ \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]\, \mathbb{1}_{\bar{Q}_{n,N}(T_n)>K} - \\
&\qquad\qquad - K \sum_{i=n+1}^{N} \beta_i\ \bar{P}(T_n, T_i)\, \mathbb{1}_{\bar{Q}_{n,N}(T_n)>K}\Bigg|\mathcal{F}_t\Bigg] \\
&\quad = \mathbb{1}_{\tau>t} \delta\, Z \sum_{j=n+1}^{N} \bar{P}(t, T_j) \mathbb{E}^{\mathbb{T}_j}\Bigg[\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \mathbb{1}_{\bar{Q}_{n,N}(T_n)>K}\Bigg|\mathcal{F}_t\Bigg] - \\
&\qquad\qquad K\, U_{n,N}(t) \mathbb{E}^{\mathbb{U}}\Bigg[\mathbb{1}_{\bar{Q}_{n,N}(T_n)>K}\Bigg|\mathcal{F}_t\Bigg]. \qquad (99)
\end{aligned}
$$

We have already seen that we can calculate the exact value of

$$
\bar{Q}_{\lambda_i}(T_n) := \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \qquad (100)
$$

which is a quadratic form in Y_{T_n}. The price of defaultable zero coupon bond prices in the quadratic Gaussian model are log-quadratic Gaussian and therefore we can write

$$
\bar{P}(T_n, T_k) = \exp(\bar{Q}_k(T_n)) \qquad (101)
$$

where

$$
\bar{Q}_k(T_n) := \log_e(\bar{P}(T_n, T_k)) = -Y_{T_n}^{\top} C(T_n, T_k) Y_{T_n} - B(T_n, T_k)^{\top} Y_{T_n} - A(T_n, T_k).
$$

Hence in (99) the value

$$
\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n, T_j)\, \mathbb{1}_{\bar{Q}_{n,N}(T_n)>K}\ \mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] \qquad (102)
$$

is a sum of payoffs of type

$$
G_3(x, \tilde{K}) := b_2.x\, \mathbb{1}_{b_0.x>K}, \quad x \in \mathbb{R}^n \qquad (103)
$$

where b_0 and b_2 are appropriately chosen. In particular

$$
\begin{aligned}
b_0 = (0, 1),\, b_2 = (1, 0),\\
x = (x_1, x_2) = (\bar{Q}_{\lambda_i}(T_n), \bar{Q}_{n,N}(T_n)).
\end{aligned}
$$

Since we know the characteristic function of a quadratic form in Gaussian random variables under $\bar{\mathbb{T}}_n$ in closed form, we can now calculate (102) using the Fourier transform method given in Lee (2004). For a payoff of type (103), Lee (2004) gives different ways of calculating the price of the corresponding option. The method is based on transforming the dampened option price with respect to the strike price K. The most efficient way of numerically inverting (107) is to use the result given in Lee (2004). For the particular case $x = (x_1, x_2)$ let us denote by $\mathcal{C}_{G_4}(K)$ the option price which is normalized by the price of the defaultable zero coupon bond with maturity equal to the maturity of the option where the payoff is of type (103). Let $\hat{\mathcal{C}}_{G_4}(z)$ denote the Fourier transform of $\mathcal{C}_{G_4}(K)$(see Lee (2004) for details)

$$\hat{\mathcal{C}}_{G_4}(z) = \frac{-b_2.\nabla\Phi(x_1, x_2, b_0 z)}{z} \tag{104}$$

where $\Phi(x_1, x_2, w_1, w_2)$ represents the joint characteristic function of two quadratic Gaussian random variables x_1, x_2 evaluated at

$$(w_1, w_2) = b_0 z.$$

The problem of obtaining the option price through Fourier inversion is given by Lee (2004) for the payoff of type (103) and it is:

$$\mathcal{C}_{G_4}(K) = R_{\hat{\alpha},G_4} + \frac{1}{\pi}\int_{0-\hat{\alpha}i}^{\infty-\hat{\alpha}i} \mathrm{Re}\left[\hat{\mathcal{C}}_{G_4}(z)\exp\left(-iz\tilde{K}\right)\right] dz \tag{105}$$

where

$$R_{\hat{\alpha},G_4} = \begin{cases} -ib_2.\nabla\Phi(x_1, x_2, 0, 0), & \hat{\alpha} < 0 \\ \dfrac{-ib_2.\nabla\Phi(x_1, x_2, 0, 0)}{2}, & \hat{\alpha} = 0 \\ 0, & \hat{\alpha} = 0 \end{cases} \tag{106}$$

The choice of the dampening factor $\hat{\alpha}$ is not restricted by the domain of existence of the characterisitc function since the characteristic function of a quadratic form in Gaussian random variables exists everywhere. Hence we can choose not to dampen the option price by choosing $\hat{\alpha} = 0$. However there are advantages in dampening the option price by different values of $\hat{\alpha}$ depending on the strike price in order to minimize the error in the Fourier inversion which is needed to calculate the price of the option(see Lee (2004) for a detailed discussion). We do not investigate the error differences obtained by choosing different $\hat{\alpha}$ for the dampening factor but use a uniform value of $\hat{\alpha} = 1$ for the different range of strike prices in our numerical experiments to be presented later in this section. Therefore we can find the Fourier transform of the dampened value

$$\int_{-\infty}^{\infty} \exp(\hat{\alpha}K)\mathbb{E}^{\bar{\mathbb{T}}_n}\left[\delta Z \sum_{j=n+1}^{N} \bar{P}(T_n, T_j)\, \mathbb{1}_{\bar{Q}_{n,N}(T_n)>K}\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}]\middle|\mathcal{F}_t\right] \exp(izK)\, d\tilde{K} \tag{107}$$

with respect to the strike price K in closed form and numerically invert the Fourier transform using (105) and (106). The calculation of the dampened value

$$\int_{-\infty}^{\infty} \exp(\hat{\alpha}K)\mathbb{E}^{\mathbb{U}}\left[\mathbb{1}_{\bar{Q}_{n,N}(T_n)>K}\middle|\mathcal{F}_t\right] \exp(izK)\, d\tilde{K} \tag{108}$$

involves a payoff of type

$$G_3(x, \tilde{K}) := \exp(b_1.x)\, \mathbb{1}_{b_0.x>K}, \quad x \in \mathbb{R}^n \tag{109}$$

where

$$b_0 = (0, 1), b_1 = (0, 0), x = (0, \bar{Q}_{n,N}(T_n)).$$

We have already discussed in Assefa (2007)(or see Lee (2004)) in the context of the approximation of the price of default free swaptions how to calculate option prices involving payoffs of type (109). Thus we have shown how to calculate the numerical inversion of the Fourier transform with respect to the strike price K of (99) which is formally given by:

$$\widehat{\widetilde{CDS}2}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)(z) = \int_{-\infty}^{\infty} \exp(izK) \exp(\hat{\alpha}K)\, \widetilde{CDS}2_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)\, d\tilde{K}. \tag{110}$$

The approximation of the price of the credit default swaption given by (99) is given by the numerical Fourier inversion and removal of the dampening[9]

[9]For a detailed discussion see Lee (2004)

through the following formula:

$$\widetilde{CDS}2_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \int_{0-\hat{\alpha}I}^{\infty-\hat{\alpha}I} \mathrm{Re}[\widehat{\widetilde{CDS}2}_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z)(z) \exp(-i\tilde{K}z)]\, dz. \quad (111)$$

We have discussed above how we can calculate (111) through the numerical inversion of several Fourier transforms and therefore we can now claim that we can calculate (99) in closed form[10]. Note however that we have to invert several Fourier transforms depending on the discretization level δ of the integral given in (98). Therefore this method takes more time to compute the approximate price of a credit default swaption. For numerical experiments we took a discretization level of $\delta = 0.0625$ in (78) which requires a larger number of numerical Fourier inversions. We can take $\delta = 0.25$ to obtain a price that differs from that of a finer discretization by a couple of basis points and therefore the computation can be speeded up. Moreover we believe that instead of using quadrature to do the numerical inversion of the Fourier transforms, we can use the discrete Fourier transform or the fast Fourier transform to obtain significant speed up of the inversion. Moreover as the number of factors increases to more than 2, the dimension of the multidimensional integral in (74) also increases by the same amount and cubature methods are slower while the approximation (99) can still be implemented efficiently through the numerical Fourier inversion. From numerical experiments given later in this section, we can see that the implementation of (99) through the Fourier technique discussed is much more accurate than (91) especially when the maturity of the credit default swaption is far from the present date $t = 0$.

We now give a third approximation for the exact price of a credit default swaption as given in (71) by using the formulation given in (91). Instead of making the assumption that we can approximate (90) by a quadratic form in a Gaussian random vector such that the Gaussian random vector has a mean and variance-covariance matrix equal to the exact mean and variance-

[10]By closed form, we mean up to a numerical inversion of the closed form Fourier transform of the option price which can be done efficiently.

covariance matrix of Y_{T_n} under $\mathbb{U}$, we calculate

$$CDS_{OP}(t, T_n, \mathcal{T}_{n,N}, T, K, Z) = \\ U_{n,N}(t)\, \mathbb{E}^{\mathbb{U}}\left[\left(\delta Z \sum_{j=n+1}^{N} \bar{w}_j(0)\mathbb{E}^{\bar{\mathbb{T}}_j}[\lambda_{T_j}|\mathcal{F}_{T_n}] - K\right)^+ \Bigg| \mathcal{F}_t\right]. \quad (112)$$

using a Gram Charlier series(see Tanaka, Yamada & Watanabe (2005)). The Gram Charlier series is used to approximate the density of (90). Therefore we calculate the exact higher moments of (90) as in (87) through a weighted sum as follows

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{U}}[(\bar{Q}_\lambda(T_n))^k] &= \mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\sum_{i=n+1}^{N} \beta_i \bar{P}(T_n, T_i)}{\sum_{i=n+1}^{N} \beta_i \frac{\bar{P}(t,T_i)}{\bar{P}(t,T_n)}} (\bar{Q}_\lambda(T_n))^k\right] \\
&= \sum_{i=n+1}^{N} \frac{\beta_i \bar{P}(t, T_i)}{\sum_{i=n+1}^{N} \beta_i \bar{P}(t, T_i)} \mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n, T_i)}{\frac{\bar{P}(t,T_i)}{\bar{P}(t,T_n)}} (\bar{Q}_\lambda(T_n))^k\right] \\
&= \sum_{i=n+1}^{N} \frac{\beta_i \bar{P}(t, T_i)}{\sum_{i=n+1}^{N} \beta_i \bar{P}(t, T_i)} \mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n, T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n, T_i)]} (\bar{Q}_\lambda(T_n))^k\right]
\end{aligned} \quad (113)$$

where we can calculate

$$\mathbb{E}_t^{\bar{\mathbb{T}}_n}\left[\frac{\bar{P}(T_n, T_i)}{\mathbb{E}^{\bar{\mathbb{T}}_n}[\bar{P}(T_n, T_i)]} (\bar{Q}_\lambda(T_n))^k\right] \quad (114)$$

by applying Lemma 5.3 of Assefa (2007) and the discussion given in Mathai and Provost (1992) regarding the efficient calculation of higher moments of quadratic forms in Gaussian random variables. Note that (114) corresponds to finding the higher moments of (90) under $\bar{\mathbb{T}}_i$ which is the defaultable forward measure[11] for maturity T_i. Even though (90) is not a quadratic form under $\mathbb{U}$, it is a quadratic form under each forward measure $\bar{\mathbb{T}}_i$ for $i = n+1, \ldots, N$.

We now present numerical results for the different approximations given in this section. We assume the default free discount and CDS data is as

[11] We use $\bar{\mathbb{T}}_i$ to denote $\bar{\mathbb{T}}_{T_i}$ to lighten the notation.

T	$P(0,T)$
11	0.451034
12	0.410806
13	0.362682
14	0.302714
15	0.226954

Table 4: Zero Rates for years 10 to 15

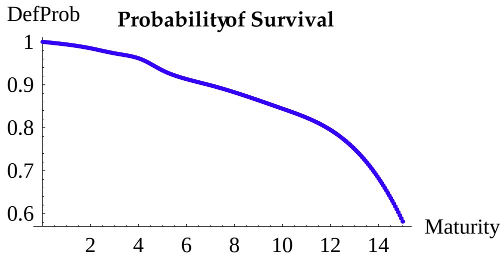

Figure 0.3.5: Extracted Survival Probability with Extrapolation for years 10 to 15

given in Tables 1 and 2. Since we would like to test the performance of the approximations for the price of credit default swaptions with maturities up to five years with an underlying CDS of maturity that can be ten years, we extended the discount data given by Table 1 through extrapolation. We give this additional default free discount data in Table 4. We extracted the probability of survival under the default free forward measure $\mathbb{T}$ which we denoted by $\bar{G}(0,T)$ as described in the previous section(see Figure 0.2.3). The survival probabilities have to be also extended by extrapolation for years 10 to 15 since we extracted the survival probability based on ten years default free discount data and CDS quotes of maximum maturity ten years. We give the figure for the extrapolated $\bar{G}(0,T)$ in Figure 0.3.5. As in the previous section we assume a two factor quadratic Gaussian model(see (66) and use the parameters given by:

$$a_{11} = 0.01, a_{22} = 0.03, \sigma_{11} = 0.02, \sigma_{22} = 0.04, \rho = 0.9. \qquad (115)$$

There is no particular reason we use these parameters, our objective is to test the performance of the approximations for these parameters. The exact value of the price of the credit default swaption is calculated based on the double integral given in (74). The maturities for the credit default swaption are given in the rows and range from $T = 1$ to $T = 5$ years. The tenor of the CDS underlying the credit default swaption range from 1 to 10 years and the price of the corresponding credit default swaption are given along the columns. For each maturity tenor pair, we give next to the exact value the value obtained through the approximation (91) in parentheses and below it the relative error for the approximation expressed as a percentage. For each of the maturity tenor pair, we consider three strike prices. The first strike price is the at the money strike price which is the strike price that would make the value of a forward starting CDS equal to zero and is obtained by solving for K in

$$\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_0^{T_n} r_s+\lambda_s\,ds\Big)\Big(\delta Z\sum_{j=1}^{N}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_{T_n}^{T_j} r_s+\lambda_s\,ds\Big)\lambda_{T_j}|\mathcal{F}_{T_n}\Big]- - K\sum_{i=1}^{N}\beta_i\,\bar{P}(T_n,T_i)\Big)\Big] = 0. \quad (116)$$

Therefore the at the money strike price for the credit default swaption of maturity T_n with an underlying CDS of tenor equal to $T_N - T_n$ years is given by

$$K_{DATM} := \frac{\delta Z\sum_{j=1}^{N}\bar{P}(0,T_j)\mathbb{E}^{\bar{\mathbb{T}}_n}[\lambda_{T_j}]}{\sum_{i=1}^{N}\beta_i\bar{P}(0,T_i)}. \quad (117)$$

We then consider an in the money strike which is taken to be $0.85 \times K_{DATM}$ and an out of the money strike which is taken to be $1.15 \times K_{DATM}$.

Tables 5,6 and 7 give the results for the approximation of the price of credit default swaptions based on (91). As we can see from these results the approximation has an error of a few basis points when the maturity of the credit default swaption is one year. For the five year maturity credit default swaption, we have an error that increases with the tenor of the underlying CDS such that for a CDS of tenor length equal to 10 years, we have a large error. There are two assumptions that could be the cause for this error. The first possible cause for the error is that we approximate Y_t by a Gaussian process with the same mean and variance-covariance matrix when deriving the approximation (91). The second possible reason is that we replace the weights $\bar{w}_j(t)$ by their time zero values. The first error can be eliminated by using the exact characteristic function of (90) as it can be calculated as the weighted sum of the characteristic functions of Gaussian quadratic forms similar to what we did in (113). Numerical tests using the exact characteristic function of (90) had in general more error in the approximate prices of the credit default swaptions obtained in comparison to the first approximation. We therefore can conclude that the error is mainly due to the fact that we replaced the weights $\bar{w}_j(T_n)$ in (90) by their time zero values. It appears replacing Y_t which has a non-Gaussian dynamics under $\mathbb{U}$ by a Gaussian process with the same mean and variance-covariance matrix has the effect of canceling some part of the error introduced by the freezing of the weights $\bar{w}_i(t)$. This observation is also supported by the results given in Tables 11, 12 and 13 which is obtained from the approximation based on a Gram Charlier series which does not replace Y_t by a Gaussian process but calculates the exact moments of (90) using (113). Since the only assumption when calculating the higher moments is that the weights $\bar{w}_j(t)$ are frozen, the source of error can come from this assumption or from the lack of accuracy by the Gram Charlier series in approximating the probability density of (90). The more likely source of error is the freezing of the weights as the Gram Charlier series method does not generally have a lot of error when a limited number of moments are used. This shows that freezing of the weights $\bar{w}(t)$ by replacing them by their time zero values leads to large errors for the prices of credit default swaptions when the underlying CDS tenors are greater or equal to five years. The results of Tables 8, 9 and 10 which are based on the approximation given by (99) are very accurate as we only approximate the exercise region of credit default swaption.

Mat.	Tenor			
T	1	3	5	10
1	17.18(16.76) (2.49%)	49.64(48.42) (2.46%)	92.16(91.64) (0.57%)	143.36(143.77) (-0.29%)
3	26.45(25.34) (4.19%)	102.78(98.9) (3.82%)	144.71(137.68) (4.86%)	238.14(242.68) (-1.91%)
5	42.67(39.72) (6.91%)	99.39(92.45) (6.98%)	152.24(143.42) (5.79%)	258.70(325.53) (-25.83%)

Table 5: Relative Error of Approximation given by (91) for $K = K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	20.85(20.42) (2.06%)	62.31(61.10) (1.96%)	122.54(122.81) (-0.22%)	202.53(202.88) (-0.18%)
3	30.00(28.89) (3.70%)	122.08(118.15) (3.22%)	172.15(165.10) (4.10%)	296.47(301.50) (-1.70%)
5	48.51(45.55) (6.09%)	112.81(105.85) (6.17%)	174.42(165.60) (5.06%)	335.02(403.07) (-20.31%)

Table 6: Relative Error of Approximation given by (91) for $K = K_{DITM} = 0.85 \times K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	14.09(13.67) (2.99%)	39.28(38.08) (3.05%)	67.90(67.37) (0.79%)	99.64(99.93) (-0.29%)
3	23.33(22.23) (4.71%)	86.37(82.49) (4.49%)	121.54(114.60) (5.71%)	190.58(194.69) (-2.16%)
5	37.54(34.61) (7.79%)	87.67(80.79) (7.84%)	133.05(124.29) (6.58%)	199.18(262.17) (-31.63%)

Table 7: Relative Error of Approximation given by (91) for $K = K_{DOTM} = 1.15 \times K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	17.18(17.15) (0.22%)	49.64(49.94) (-0.61%)	92.16(92.18) (-0.02%)	143.36(143.09) (0.18%)
3	26.45(26.41) (0.15%)	102.78(103.02) (-0.23%)	144.71(145.39) (-0.46%)	238.14(238.03) (0.05%)
5	42.67(42.51) (0.36%)	99.39(99.85) (-0.46%)	152.24(151.93) (0.20%)	258.70(258.17) (0.21%)

Table 8: Relative Error of Approximation given by (99) for $K = K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	20.85(20.81) (0.18%)	62.31(62.33) (-0.02%)	122.54(123.32) (-0.64%)	202.53(202.44) (0.04%)
3	30.00(30.11) (-0.37%)	122.08(121.17) (0.75%)	172.15(172.36) (-0.12%)	296.47(296.78) (-0.10%)
5	48.51(48.25) (0.54%)	112.81(112.71) (0.09%)	174.42(174.38) (0.03%)	335.02(334.70) (0.09%)

Table 9: Relative Error of Approximation given by (99) for $K = K_{DITM} = 0.85 \times K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	14.09(14.06) (0.26%)	39.28(39.16) (0.31%)	67.90(67.71) (0.28%)	99.65(99.73) (-0.08%)
3	23.33(23.41) (-0.33%)	86.37(86.29) (0.09%)	121.54(123.17) (-1.34%)	190.58(190.63) (-0.02%)
5	37.54(37.44) (0.26%)	87.67(87.87) (-0.23%)	133.05(133.30) (-0.19%)	199.18(197.96) (0.61%)

Table 10: Relative Error of Approximation given by (99) for $K = K_{DOTM} = 1.15 \times K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	17.18(17.29) (-0.57%)	49.64(49.95) (-0.62%)	92.16(93.71) (-1.68%)	143.36(147.74) (-3.06%)
3	26.45(27.64) (-4.49%)	102.78(103.97) (-1.16%)	144.71(146.75) (-1.41%)	238.14(258.72) (-8.64%)
5	42.67(42.72) (-0.12%)	99.39(102.23) (-2.86%)	152.24(159.18) (-4.56%)	258.70(351.12) (-35.72%)

Table 11: Relative Error of Approximation based on a Gram Charlier series(based on 3 moments) for $K = K_{DATM}$

Mat.	Tenor			
T	1	3	5	10
1	20.85(21.00) (-0.70%)	62.31(62.81) (-0.79%)	122.54(125.21) (-2.17%)	202.53(207.62) (-2.51%)
3	30.00(31.29) (-4.32%)	122.08(123.69) (-1.32%)	172.15(174.90) (-1.60%)	296.47(319.08) (-7.63%)
5	48.51(48.71) (-0.42%)	112.81(116.03) (-2.85%)	174.42(182.03) (-4.36%)	335.02(431.02) (-28.68%)

Table 12: Relative Error of Approximation based on a Gram Charlier series(based on 3 moments)for $K = K_{DITM} = 0.85 \times K_{DATM}$

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Mat.	Tenor			
T	1	3	5	10
1	14.09(14.20) (-0.75%)	39.28(39.63) (-0.88%)	67.90(69.51) (-2.37%)	99.65(104.07) (-4.44%)
3	23.33(24.50) (-4.99%)	86.37(87.55) (-1.37%)	121.54(123.57) (-1.67%)	190.58(210.58) (-10.49%)
5	37.54(37.57) (-0.09%)	87.67(90.49) (-3.22%)	133.05(139.95) (-5.18%)	199.17(287.19) (-44.19%)

Table 13: Relative Error of Approximation based on a Gram Charlier series(based on 3 moments) for $K = K_{DOTM} = 1.15 \times K_{DATM}$

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