Modelling of Successive Default Events

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1 Introduction

[1] In the credit risk analysis, the dependence of default times is one of most important issues, for the portfolio credit derivatives as basket default swaps and CDOs, and also for the contagious credit risks. In the literature, the modelling of multi credit names is diversified in various directions such as Markov models ([3, 4]), contagion models ([10]), latent variable models ([8]) and loss process models ([5], [9], [11]), etc.

In this paper, we propose a new method to study credit dependence. Our aim is firstly to propose a dynamic pricing model for the portfolio credit derivatives, and secondly to make clear the impact of one default event on the other ones. The methodology is based on the default density approach introduced in [7] which is suitable for the after-default analysis. We shall concentrate on successive default events, where the before and after default studies adapt naturally. Moreover, this viewpoint allows us to include individual default time information and to obtain, by using a recursive procedure, pricing formulas for k^{th}-to-default swaps and CDOs.

In this context of multi credit names, the market information becomes complicated since it concerns filtrations associated to different stopping times, besides the "default-free" reference filtration $\mathbb{F}$. The pricing problem consists of computing conditional expectations with respect to the market filtration $\mathbb{G}$, which is the enlarged filtration of $\mathbb{F}$ and all default filtrations. It

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[1]This is a preliminary working version.

should be noted that for a portfolio of size n, there exist 2^n possible default events. However, if we consider the ordered set of default times, we limit ourselves to $n+1$ default scenarios, which reduces largely the computation burden. By studying ordered defaults, we do not distinguish specific credit names. This covers partial market information, which is however sufficient for most credit portfolio products, notably k^{th}-to-default swaps and CDOs. The main idea is to apply, in a recursive manner, the "before-default" and "after-default" results developed in [7] to the ordered default times and to establish the relationship between $\mathbb{G}$-conditional expectations and $\mathbb{F}$-conditional expectations. This is done under the key hypothesis that the joint conditional survival probabilities of the ordered default times admit a density given $\mathbb{F}$. We are also interested in the characterization of $\mathbb{G}$-martingales in terms of $\mathbb{F}$-(local) martingales and are capable to deduce intensity processes of each (ordered) default time. From these explicit results, we analyze the impact of the first default on the second one such as the conditional survival probability given that the first default occurs, and the contagious jump of the intensity at the first default time etc.

This paper is organized as follows: we recall in Section 2 some useful results established in [7]. In Section 3, we explain how to apply the density framework to two default times. We make precise the joint density hypothesis and deduce from it the marginal densities of the first and the second default. We then compute the conditional expectations on the three sets: before first default, between two defaults and after second default; discuss the H-hypothesis in this setting and deduce the intensity processes. The martingale characterization results are then presented. At the end of this section, we give several examples of the joint density. Section 4 is dedicated to the generalization of this methodology for an n-sized portfolio and to the pricing of k^{th}-to default swaps and CDOs. We conclude finally in the last section.

2 Default density framework

We briefly recall the density approach and some results developed in [7] which will be useful in this paper. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and τ be a finite random time. Let $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ be a reference filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ which satisfies the usual conditions. We assume that the $\mathbb{F}$-conditional distribution of τ admits a density with respect to the Lebesgue measure [2], i.e., for any $t \geq 0$, there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to \alpha_t(\omega, \theta)$ which satisfies

$$\mathbb{P}(\tau \in d\theta | \mathcal{F}_t) := \alpha_t(\theta) d\theta, \quad \mathbb{P} - a.s.. \tag{1}$$

[2] this can be generalized to the case of a non-negative non-atomic measure ad done in [7]

We call $\alpha(\theta)$ the $\mathbb{F}$-density of τ. Then the conditional survival probability is given by $S_t(\theta) := \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)du$. Furthermore, for any $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable $Y_t(\tau)$, $\mathbb{E}[Y_t(\tau)|\mathcal{F}_t] = \int_0^\infty Y_t(u)\alpha_t(u)du$.

2.1 Computation of conditional expectations

We specify the default information by the filtration $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$, which is the smallest right-continuous filtration such that τ is a $\mathbb{D}$-stopping time. We define the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$ as the global market information, which is the smallest filtration containing $\mathbb{F}$ and making τ a stopping time.

Theorem 2.1 *Let $Y_T(\tau)$ be a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-r.v.. Then, for any $t \leq T$,*

$$E[Y_T(\tau)|\mathcal{G}_t] = \frac{\mathbb{E}\big[\int_t^\infty Y_T(u)\alpha_T(u)du|\mathcal{F}_t\big]}{S_t} 1_{\{t<\tau\}} + Y_t^{\rm ad}(T,\tau)1_{\{\tau\leq t\}} \quad d\mathbb{P} - a.s. \tag{2}$$

where $S_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ and

$$Y_t^{\rm ad}(T,\theta) = \frac{\mathbb{E}\big[Y_T(\theta)\alpha_T(\theta)\big|\mathcal{F}_t\big]}{\alpha_t(\theta)}. \tag{3}$$

We observe that the $\mathbb{G}$ conditional expectations are deduced on the two sets: before-default $\{t < \tau\}$ and after-default $\{\tau \leq t\}$, respectively as $\mathbb{F}$ conditional expectations.

2.2 Density and intensity

There exists an explicit relationship between the $\mathbb{F}$-density and the $\mathbb{G}$-intensity of τ. Recall that the $\mathbb{G}$-intensity of τ is the positive $\mathbb{G}$-adapted process $\lambda^{\mathbb{G}}$ such that $(1_{\{\tau\leq t\}} - \int_0^t \lambda_s^{\mathbb{G}} ds, t \geq 0)$ is a $\mathbb{G}$-martingale.

Proposition 2.2
1) The $\mathbb{G}$-intensity of τ is given by

$$\lambda_t^{\mathbb{G}} = 1_{\{\tau>t\}} \frac{\alpha_t(t)}{S_t}. \tag{4}$$

2) For any $\theta \geq t$, we have $\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{G}}|\mathcal{F}_t]$.

The above proposition shows us that $\lambda_t^{\mathbb{G}}$ can be completely deduced from $\alpha_t(t)$ since $S_t = \int_t^\infty \alpha_t(\theta)d\theta = \int_t^\infty \mathbb{E}(\alpha_\theta(\theta)|\mathcal{F}_t)d\theta$. However, given $\lambda_t^{\mathbb{G}}$, we can only obtain partial knowledge of

$\alpha_t(\theta)$ for $\theta \geq t$. This is the reason why the intensity approach is not sufficient to study what goes on after a default event, since we observe clearly that in (3), $\alpha_t(\theta)$ where $\theta \leq t$ play an important role.

2.3 Martingale characterization

The $\mathbb{G}$-martingales can be characterized in terms of $\mathbb{F}$-(local) martingales using the $\mathbb{F}$-density. In the following theorem, the first condition is for the $\mathbb{G}$-martingales which are stopped at τ and the second one is for those which start at τ.

Theorem 2.3 *A càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that $Y_t^{\mathbb{G}} = Y_t 1_{\{\tau > t\}} + Y_t(\tau) 1_{\{\tau \leq t\}}$ and that*
1) $(Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(ds),\, t \geq 0)$ is an $\mathbb{F}$-local martingale;
2) for any $\theta \geq 0$, $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale.

3 Two ordered default times

In this section, we consider two (ordered) default times and we explain how to generalize the density framework in the multi-defaults setting.

3.1 Setup and notation

We first introduce some notation. Let $(\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and τ_1 and τ_2 be two random times on $(\Omega, \mathcal{A}, \mathbb{P})$. Let us consider the ordered default times

$$\tau = \tau^{(1)} := \min(\tau_1, \tau_2) \quad \text{and} \quad \sigma = \tau^{(2)} := \max(\tau_1, \tau_2).$$

We call τ the first default and σ the second one. Let $\mathbb{D}^{(1)}$ and $\mathbb{D}^{(2)}$ be the associated filtrations of τ and σ respectively. Let $\mathbb{G}^{(1)} = \mathbb{F} \vee \mathbb{D}^{(1)}$ and $\mathbb{G}^{(2)} = \mathbb{F} \vee \mathbb{D}^{(1)} \vee \mathbb{D}^{(2)}$.

The before-default and after-default analysis with density can be naturally extended to ordered default times. We now introduce the density hypothesis for (τ, σ), assuming the existence of the joint conditional density.

Hypothesis 3.1 (two defaults density hypothesis) Assume that, for any $t \in \mathbb{R}_+$, the conditional distribution of (τ, σ) given $\mathcal{F}_t$ is absolutely continuous with respect to the Lebesgue measure. That is, for any $t \geq 0$, there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^2)$-measurable random variables $\alpha_t(u, v)$ such that

$$\mathbb{P}(\tau \in du, \sigma \in dv | \mathcal{F}_t) = \alpha_t(u, v) du dv, \quad \mathbb{P} - a.s.. \tag{5}$$

Or, in other words [3], $S_t(\theta_1, \theta_2) := \mathbb{P}(\tau > \theta_1, \sigma > \theta_2 | \mathcal{F}_t) = \int_{\theta_1}^{\infty} \int_{\theta_2}^{\infty} \alpha_t(u, v) du dv.$

Note that

$$\alpha_t(u, v) = 0, \quad \forall u > v. \tag{6}$$

From the dynamic point of view, there exists a universal cadlag version of the $\mathbb{F}$-martingale density. We still use the same notation to denote the density process. In the following of this section, we always suppose Hypothesis 3.1.

The methodology is now simple. We shall use the results established in [7] with various choices of reference filtration

- before default τ, i.e., on the set $\{t < \tau\}$, , with reference filtration $\mathbb{F}$;
- before default σ, i.e., on the set $\{\tau \leq t < \sigma\}$, with reference filtration $\mathbb{G}^{(1)}$;
- after default σ, i.e., on the set $\{\sigma < t\}$, with reference filtration $\mathbb{G}^{(1)}$.

We firstly give the $\mathbb{F}$-density of τ using the joint $\mathbb{F}$-density of (τ, σ).

Lemma 3.2 *The survival distribution of τ w.r.t $\mathbb{F}$ is given by*

$$S_t^1(\theta) := \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_{\theta}^{\infty} \int_{u}^{\infty} \alpha_t(u, v) du \, dv \, .$$

The $\mathbb{F}$-density of τ is given by [4]

$$\alpha_t^1(\theta) = \int_{\theta}^{\infty} \alpha_t(\theta, v) dv, \quad a.s.. \tag{7}$$

Proof. It suffices to observe that

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \mathbb{P}(\tau > \theta, \sigma > \theta | \mathcal{F}_t) = S_t(\theta, \theta) = \int_{\theta}^{\infty} \int_{\theta}^{\infty} \alpha_t(u, v) du \, dv$$

and then use (6). The $\mathbb{F}$-density of τ is given by $\alpha_t^1(\theta) = \mathbb{P}(\tau \in d\theta | \mathcal{F}_t)$. □

[3] This can be generalized to the case $\mathbb{P}(\tau > \theta_1, \sigma > \theta_2 | \mathcal{F}_t) = \int_{\theta_1}^{\infty} \int_{\theta_2}^{\infty} \alpha_t(u, v) \eta_1(du) \eta_2(dv)$, where η_1 and η_2 are non-negative non-atomic measures.

[4] We have chosen concise notation. More precise ones would be $S_t^1(\theta) = S_t^{\tau|\mathbb{F}}(\theta)$, and $\alpha_t^1(\theta) = \alpha_t^{\tau|\mathbb{F}}(\theta)$. For only two defaults, there is no ambiguity.

3.2 $\mathbb{G}^{(1)}$-Density of second default σ

We now concentrate on the second default σ. As $\mathbb{G}^{(2)} = \mathbb{G}^{(1)} \vee \mathbb{D}^{(2)}$, one can apply results developed in [7] with reference filtration $\mathbb{G}^{(1)}$. We begin by giving the $\mathbb{G}^{(1)}$-density of σ, i.e., $\alpha_t^2(\theta) := \mathbb{P}(\sigma \in d\theta | \mathcal{G}_t^{(1)})$ for any $\theta \in \mathbb{R}_+$.

Proposition 3.3 *The $\mathbb{G}^{(1)}$-density of σ is given by*

$$\alpha_t^2(\theta) = 1_{\{\tau>t\}} \frac{\int_t^\infty \alpha_t(u,\theta)du}{S_t^1(t)} + 1_{\{\tau\leq t\}} \frac{\alpha_t(\tau,\theta)}{\alpha_t^1(\tau)}, \quad a.s.. \tag{8}$$

If $\theta < t$, then by (6), the first term vanishes, and

$$\alpha_t^2(\theta) = 1_{\{\tau\leq t\}} \frac{\alpha_t(\tau,\theta)}{\alpha_t^1(\tau)}, \quad \forall \theta < t.$$

If in addition, $\theta < \min(t,\tau)$, then $\alpha_t^2(\theta) = 0$.

Proof. On the set $\{\tau > t\}$, we compute the quantity $S_t^2(\theta) := \mathbb{P}(\sigma > \theta | \mathcal{G}_t^{(1)})$,

$$S_t^2(\theta) 1_{\{t<\tau\}} = \frac{\mathbb{P}(\tau > t, \sigma > \theta | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)} 1_{\{t<\tau\}} = 1_{\{t<\tau\}} \frac{\int_t^\infty \int_\theta^\infty \alpha_t(u,v)du\,dv}{S_t^1(t)}. \tag{9}$$

Differentiating w.r.t. θ leads the result.
On the set $\{\tau \leq t\}$, the $\mathbb{F}$-conditional density of σ given τ is obtained directly as the quotient between the $\mathbb{F}$-density of (τ,σ) and of τ evaluated at τ. □

Remark 3.4 We may examine the impact of the first default τ by taking a closer look at the $\mathbb{G}^{(1)}$-survival probability and density of σ. Before τ, i.e. on the set $\{t < \tau\}$, the σ-algebra $\mathcal{F}_t$ represents the information accessible to investors and the survival probability based on this information is $\mathbb{P}(\sigma > \theta|\mathcal{F}_t)$. When we take into account the fact that $t < \tau$, then (9) can be viewed as the $\mathbb{F}$-survival probability conditional on this event $\{\tau > t\}$. After τ, we shall add this new information of τ to $\mathcal{F}_t$. Indeed, we observe from (8) that the $\mathbb{G}^{(1)}$-density of σ is nothing but the conditional density of σ given $\mathcal{F}_t$ and τ.

As seen in the above proof, the $\mathbb{G}^{(1)}$-survival process of σ is the supermartingale given by

$$S_t^2 := \mathbb{P}(\sigma > t | \mathcal{G}_t^{(1)}) = 1_{\{\tau>t\}} + 1_{\{\tau\leq t\}} \frac{\int_t^\infty \alpha_t(\tau,v)dv}{\alpha_t^1(\tau)}, \tag{10}$$

with which we can deduce the intensity of the second default σ by using (4).

Proposition 3.5 *The* $\mathbb{G}^{(2)}$*-intensity of* σ *is given by*

$$\lambda_t^{\sigma,\mathbb{G}^{(2)}} = 1_{\{\tau\le t<\sigma\}} \frac{\alpha_t(\tau,t)}{\int_t^\infty \alpha_t(\tau,v)dv}. \tag{11}$$

Proof. By Proposition 2.2,

$$\lambda_t^{\sigma,\mathbb{G}^{(2)}} = 1_{\{\sigma>t\}} \frac{\alpha_t^2(t)}{S_t^2}.$$

Then by using the explicit forms of $\mathbb{G}^{(1)}$-density and survival process (8) and (10) of σ, we have $\alpha_t^2(t) = 0$ and hence $\lambda_t^{\sigma,\mathbb{G}^{(2)}} = 0$ for any $t < \tau$ and deduce the equality (11). □

Remark 3.6 Before τ, the intensity of the second default σ remains zero. At time τ, the first default has an impact and the intensity of σ has a positive jump $\lambda_\tau^{\sigma,\mathbb{G}^{(2)}}$.

Remark 3.7 The notion "intensity" is related to a filtration. For the first default τ, Proposition 2.2 gives its $\mathbb{G}^{(1)}$-intensity in terms of $\mathbb{F}$-intensity. In addition, since the filtrations $\mathbb{G}^{(1)}$, $\mathbb{G}^{(2)}$ and $\mathbb{G}$ coincide on $\{\tau > t\}$, the intensities of τ w.r.t. $\mathbb{G}^{(1)}$, $\mathbb{G}^{(2)}$ and $\mathbb{G}$ also coincide. This means that the $\mathbb{G}^{(1)}$-martingale $(1_{\{\tau\le t\}} - \int_0^t \lambda_t^{\tau,\mathbb{G}^{(1)}} dt, t \ge 0)$, which is stopped at τ, is also a martingale w.r.t. $\mathbb{G}^{(2)}$ and $\mathbb{G}$. Similarly, for the second default σ, its intensities w.r.t. $\mathbb{G}^{(2)}$ and $\mathbb{G}$ coincide.

3.3 Computation of conditional expectations

We now compute $\mathbb{G}^{(2)}$ conditional expectations by applying (2). As we said before, the analysis has to be done on the sets $\{t < \tau\}$, $\{\tau \le t < \sigma\}$ and $\{\sigma \le t\}$, which correspond to the periods before the first default, between the two defaults and after the second default. By using the density process, the conditional expectations reduce to integration formulas.

We first introduce some notation to simplify the formulas in further computations. Let t be fixed, and consider a map $(\omega, t_1, t_2) \to Y_t(t_1, t_2; \omega)$ which is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^2)$ measurable, and the associated random variable $Y_t(\tau, \sigma)$. As in the single-name case, and taking care of $\alpha_t(u, v) = 0$ for $v < u$,

$$\alpha_t(Y_t) := \mathbb{E}[Y_t(\tau,\sigma)|\mathcal{F}_t] = \int_0^\infty du \int_u^\infty dv \alpha_t(u,v) Y_t(u,v).$$

For any pair $T, t \ge 0$, we define

$$\alpha_{T,t}(Y_T) := \int_t^\infty du \int_u^\infty dv Y_T(u,v)\alpha_T(u,v); \tag{12}$$

$$\alpha_{T,t}^{(u)}(Y_T) := \int_t^\infty dv Y_T(u,v)\alpha_T(u,v), \quad \alpha_{T,t}^{(v)}(Y_T) := \int_t^\infty du Y_T(u,v)\alpha_T(u,v); \tag{13}$$

$$\alpha_t^{(u,v)}(Y_t) := Y_t(u,v)\alpha_t(u,v). \tag{14}$$

The computation on the set $\{\tau > t\}$ is obtained noting that, on this set, $\mathbb{G}$-adapted processes are equal to processes adapted w.r.t. the original reference filtration $\mathbb{F}$. The set $\{\tau \le t < \sigma\}$ corresponds to the before-default case of σ and the after-default case of τ. We shall apply a recursive procedure: firstly we study σ with respect to the new reference filtration $\mathbb{G}^{(1)}$; secondly we use the passage between the filtrations $\mathbb{G}^{(1)}$ and $\mathbb{F}$ and the explicit form of $\mathbb{G}^{(1)}$-density $\alpha_t^2(\theta)$ of σ. Then all $\mathbb{G}^{(2)}$-conditional expectations can be deduced by using $\mathbb{F}$-conditional expectations. The set $\{\sigma \le t\}$ can be treated similarly by the recursive procedure.

We summarize in the following the computation results of $\mathbb{G}^{(2)}$-conditional expectations, which, as we shall explain further on, will give a representation of $\mathbb{G}^{(2)}$-martingales. Observe that the formulas are quite regular on different default scenario sets.

Theorem 3.8 *Let $Y_T(\tau,\sigma)$ be a random variable where $\omega, t_1, t_2 \to Y_T(t_1,t_2;\omega)$ is $\mathcal{F}_T \times \mathcal{B}(\mathbb{R}^2)$ measurable. Then for any $t \le T$,*

$$\mathbb{E}\big[Y_T(\tau,\sigma)\,|\,\mathcal{G}_t^{(2)}\big] = 1_{\{\tau>t\}}\,q_t^1(T,Y_T) + 1_{\{\tau\le t<\sigma\}}\,q_t^2(T,\tau,Y_T) + 1_{\{\sigma\le t\}}\,q_t^3(T,\tau,\sigma,Y_T) \tag{15}$$

where

$$q_t^1(T,Y_T) = \frac{\mathbb{E}\big[\alpha_{T,t}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_{t,t}(1)}, \quad q_t^2(T,\tau,Y_T) = \frac{\mathbb{E}\big[\alpha_{T,t}^{(u)}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_{t,t}^{(u)}(1)}\Big|_{u=\tau}$$

and

$$q_t^3(T,\tau,\sigma,Y_T) = \frac{\mathbb{E}\big[\alpha_T^{(u,v)}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_t^{(u,v)}(1)}\Big|_{\substack{u=\tau\\ v=\sigma}}$$

with the notation being defined in (12), (13) *and* (14).

Proof. Firstly, noting that $S_t^1 := S_t^1(t) = \mathbb{P}(\tau > t|\mathcal{F}_t)$ is nothing but $\alpha_{t,t}(1)$, we have directly

$$\mathbb{E}\big[Y_T(\tau,\sigma)\,|\,\mathcal{G}_t^{(2)}\big]1_{\{\tau>t\}} = \frac{\mathbb{E}\big[\alpha_{T,t}(Y_T)\,|\,\mathcal{F}_t\big]}{\alpha_{t,t}(1)}\,1_{\{\tau>t\}}, \quad a.s.. \tag{16}$$

For the set $\{\tau \le t < \sigma\}$, by using (2), we obtain

$$\begin{aligned}\mathbb{E}\big[Y_T(\tau,\sigma)\,|\,\mathcal{G}_t^{(2)}\big]1_{\{\tau\le t<\sigma\}} &= \frac{\mathbb{E}\big[\int_t^\infty dv Y_T(\tau,v)1_{\{\tau\le t\}}\,\alpha_T^2(v)\,|\,\mathcal{G}_t^{(1)}\big]}{\int_t^\infty dv\,\alpha_t^2(v)}1_{\{\tau\le t<\sigma\}}\\ &= \frac{\mathbb{E}\big[\int_t^\infty dv\,Y_T(u,v)\alpha_T(u,v)|\mathcal{F}_t\big]}{\int_t^\infty dv\alpha_t(u,v)}\Big|_{u=\tau}1_{\{\tau\le t<\sigma\}},\end{aligned}$$

where the second equality above is from Theorem 2.1 and (8). For the set $\{\sigma \le t\}$, similarly, we have by Theorem 2.1 and (8) that

$$\begin{aligned}\mathbb{E}\big[Y_T(\tau,\sigma)\,|\,\mathcal{G}_t^{(2)}\big]1_{\{\sigma\le t\}} &= \frac{\mathbb{E}\big[Y_T(\tau,v)\alpha_T^2(v)\,|\,\mathcal{G}_t^{(1)}\big]}{\alpha_t^2(v)}\Big|_{v=\sigma}1_{\{\sigma\le t\}}\\ &= \frac{\mathbb{E}\big[Y_T(u,v)\alpha_T(u,v)\,|\,\mathcal{F}_t\big]}{\alpha_t(u,v)}\Big|_{\substack{u=\tau\\ v=\sigma}}1_{\{\sigma\le t\}}.\end{aligned}$$

□

Remark 3.9 In general, the process defined in (15) is not continuous. We observe in fact the "contagious jump" $q_\tau^2(T,\tau,Y_T) - q_{\tau-}^1(T,Y_T)$ once the first default occurs.

3.4 H-hypothesis

We now study the immersion property (H-hypothesis) between the filtrations, in particular, between the filtrations $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$. Immersion holds between $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ if $S_t^2(\theta) = S_\theta^2(\theta)$, or equivalently $\alpha_t^2(\theta) = \alpha_\theta^2(\theta)$ for all $\theta \le t$, similarly as in the single default case discussed in [7]. Observe that for any $\theta \le t$,

$$S_t^2(\theta) = 1_{\{\tau>\theta\}} + 1_{\{\tau\le\theta\}}\frac{\int_\theta^\infty dv\alpha_t(\tau,v)}{\int_\tau^\infty dv\alpha_t(\tau,v)}.$$

In the special case where $\mathbb{F}$ is trivial, it's clear that immersion holds between $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$. This property has been pointed out in [1]. In the general case, the H-hypothesis between $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ is a quite strong hypothesis. We give below some conditions.

Proposition 3.10 *Assume immersion holds between $\mathbb{F}$ and $\mathbb{G}^{(1)}$. In addition, if $\alpha_t(\theta_1,\theta_2) = \alpha_{\theta_2}(\theta_1,\theta_2)$ for any $\theta_1 \le \theta_2 \le t$, then immersion holds between $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$.*

Proof. We shall verify the equality between the density $\alpha_t^2(\theta)$ and $\alpha_\theta^2(\theta)$ for any $\theta \le t$. By (8), we only need to check on the set $\{\tau \le \theta \le t\}$, where

$$\alpha_t^2(\theta) = 1_{\{\tau\le t\}}\frac{\alpha_t(\tau,\theta)}{\alpha_t^1(\tau)} = 1_{\{\tau\le t\}}\frac{\alpha_\theta(\tau,\theta)}{\alpha_\theta^1(\tau)} = \alpha_{\theta_2}^2(\theta_2),$$

which ends the proof. □

It is proved in [6] that $\mathbb{F}$ is immersed in $\mathbb{G}^{(2)}$ if and only if $\mathbb{F}$ is immersed in $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(1)}$ is immersed in $\mathbb{G}^{(2)}$.

3.5 $\mathbb{G}^{(2)}$-Martingale representation

Generally speaking, any $\mathbb{G}^{(2)}$-martingale can be decomposed into three processes: one stopped at the first default τ, one starting at τ and stopped at the second default σ, and the last one starting at σ. We now generalize Theorem 2.3 to the two names setting.

Theorem 3.11 *Let $Y_t^{(2)} = Y_t 1_{\{t<\tau\}} + Y_t(\tau)1_{\{\tau\le t<\sigma\}} + Y_t(\tau,\sigma)1_{\{\sigma\le t\}}$ where Y is an $\mathbb{F}$-adapted process, $Y(\tau)$ and $Y(\tau,\sigma)$ are respectively $\mathcal{O}(\mathbb{F})\times\mathcal{B}(\mathbb{R}_+)$ and $\mathcal{O}(\mathbb{F})\times\mathcal{B}(\mathbb{R}_+^2)$ optional processes. Then the process $Y^{(2)}$ is a $\mathbb{G}^{(2)}$-martingale if and only if the three processes*

$$\begin{cases} Y_t S_t + \int_0^t Y_s(s)\alpha_s^1(s)ds, \ t\ge 0; \\ Y_t(\theta)\displaystyle\int_t^\infty \alpha_t(\theta,s)ds + \int_\theta^t Y_s(\theta,s)\alpha_s(\theta,s)ds, \ t\ge\theta; \\ Y_t(\theta_1,\theta_2)\alpha_t(\theta_1,\theta_2), \ t\ge\theta_2\ge\theta_1; \end{cases} \tag{17}$$

are $\mathbb{F}$-martingales where $S_t = \int_t^\infty\int_t^\infty du\,dv\alpha_t(u,v)$.

Proof. The proof is made in two steps. Let $\widetilde{y}_t = Y_t 1_{\{t<\tau\}} + Y_t(\tau)1_{\{\tau\le t<\sigma\}}$. By Theorem 2.3, the process $(Y_t^2 = \widetilde{y}_t 1_{\{t<\sigma\}} + Y_t(\tau,\sigma)1_{\{\sigma\le t\}}, t\ge 0)$ is a $\mathbb{G}^{(2)}$-martingale if (and only if)

$$\begin{cases} \widetilde{y}_t S_t^2 + \int_0^t Y_s(\tau,s)1_{\{\tau\le s\}}\alpha_s^2(s)ds, \ t\ge 0; \\ Y_t(\tau,\theta)\alpha_t^2(\theta)1_{\{\tau\le t\}}, \ t\ge\theta; \end{cases} \tag{18}$$

are $\mathbb{G}^{(1)}$-martingales. Then using explicit forms of $\alpha_t^2(\theta)$ and S_t^2, and replacing the quantity $\widetilde{y}_t$, the above two processes can be written as

$$\begin{cases} Y_t 1_{\{\tau>t\}} + 1_{\{\tau\le t\}}\left(Y_t(\tau)\dfrac{1}{\alpha_t^1(\tau)}\displaystyle\int_t^\infty \alpha_t(\tau,s)ds + \int_\tau^t Y_s(\tau,s)\frac{\alpha_s(\tau,s)}{\alpha_s^1(\tau)}ds \right), \ t\ge 0; \\ Y_t(\tau,\theta)\dfrac{\alpha_t(\tau,\theta)}{\alpha_t^1(\tau)}1_{\{\tau\le t\}}, \ t\ge\theta. \end{cases} \tag{19}$$

Then, it remains to note that these two processes are $\mathbb{G}^{(1)}$ martingales if (and only if) the following processes are $\mathbb{F}$-martingales:

$$\begin{cases} Y_t S_t + \displaystyle\int_0^t Y_s(s)\alpha_s^1(s)ds, \ t\ge 0; \\ \displaystyle\int_0^t Y_s(s,\theta)\alpha_s(s,\theta)ds, \ t\ge\theta; \\ Y_t(\theta)\displaystyle\int_t^\infty \alpha_t(\theta,s)ds + \alpha_t^1(\theta)\int_\theta^t Y_s(\theta,s)\frac{\alpha_s(\theta,s)}{\alpha_s^1(\theta)}ds, \ t\ge\theta; \\ Y_t(\theta_1,\theta_2)\alpha_t(\theta_1,\theta_2), \ t\ge\theta_2\ge\theta_1. \end{cases} \tag{20}$$

Now, it is clear that, for $t>\theta$,

$$\int_0^t Y_s(s,\theta)\alpha_s(s,\theta)ds = \int_0^\theta Y_s(s,\theta)\alpha_s(s,\theta)ds,$$

so the second process is an $\mathbb{F}$-martingale. Furthermore, from the $\mathbb{F}$-martingale property of $\alpha^1(\theta)$, the third condition is equivalent to

$$\left(Y_t(\theta)\int_t^\infty \alpha_t(\theta,s)ds + \int_\theta^t Y_s(\theta,s)\alpha_s(\theta,s)ds, t\geq\theta\right)$$

is an $\mathbb{F}$-martingale, which completes the proof. □

Remark 3.12 Given an $\mathcal{F}_T\times\mathcal{B}(\mathbb{R}_+^2)$ random variable $Y_T(\tau,\sigma)$, its conditional expectation w.r.t. $\mathcal{G}_t^{(2)}$ forms a $\mathbb{G}^{(2)}$-martingale which, according to Theorem 3.8, is written as the sum of three processes respectively on $\{\tau>t\}$, $\{\tau\leq t<\sigma\}$ and $\{\sigma\leq t\}$. The characterization conditions in the above theorem are clearly satisfied for this martingale.

3.6 Modelling of joint $\mathbb{F}$ density

The joint $\mathbb{F}$ density of (τ,σ) plays a crucial role in previous discussions. In the following, we propose several modelling methods. Instead of working directly on $\alpha_t(\theta_1,\theta_2)$, we shall firstly concentrate on the conditional joint probability $S_t(\theta_1,\theta_2)$. Note that the relationship between the ordered default times (τ,σ) and the non-ordered ones (τ_1,τ_2) can be established by using the statistics order. The following lemma allows us to deduce the density of (τ,σ) from the non-ordered conditional probability distribution of (τ_1,τ_2).

Lemma 3.13 *For all θ_1,θ_2 and t positive, denote by $\widetilde{S}_t(\theta_1,\theta_2)$ the conditional distribution of (τ_1,τ_2) given $\mathcal{F}_t$, i.e. $\widetilde{S}_t(\theta_1,\theta_2)=\mathbb{P}(\tau_1>\theta_1,\tau_2>\theta_2|\mathcal{F}_t)$. Then*

$$S_t(\theta_1,\theta_2)=\widetilde{S}_t(\theta_1,\theta_2)+\widetilde{S}_t(\theta_2,\theta_1).$$

If, in addition, $\widetilde{S}_t(\theta_1,\theta_2)$ admits a density $\widetilde{\alpha}_t(\theta_1,\theta_2)$, then

$$\alpha_t(\theta_1,\theta_2)=1_{\{\theta_1\leq\theta_2\}}\big(\widetilde{\alpha}_t(\theta_1,\theta_2)+\widetilde{\alpha}_t(\theta_2,\theta_1)\big).$$

3.6.1 A backward example

The following example is based on the Cox process model. The idea comes from [12] that τ_1 and τ_2 are supposed to be correlated by a copula function given $\mathcal{F}_\infty$.

Example 3.14 Let τ_1 and τ_2 be defined as in the Cox process model. That is, $\tau_i=\inf\{t:\Phi_t^i\geq\xi_i\}$ $(i=1,2)$ where Φ^i is an $\mathbb{F}$-adapted increasing process with $\Phi_0^i=0$ and $\lim_{t\to\infty}\Phi_t^i=+\infty$, ξ_i is a $\mathcal{A}$-measurable random variable which follows exponential law with parameter 1 and is independent of $\mathcal{F}_\infty$. In this model, the marginal survival process is given by $\widetilde{S}^i_t = \mathbb{P}(\tau_i > t|\mathcal{F}_\infty) = e^{-\Phi^i_t}$. So the (H)-hypothesis holds between $\mathbb{F}$ and $\mathbb{G}^i = \mathbb{F} \vee \mathbb{D}^i$. Let the correlation of defaults be represented by a copula function $\mathcal{C}(x_1, x_2)$ such that $\mathbb{P}(\tau_1 > u, \tau_2 > v|\mathcal{F}_\infty) = \mathcal{C}(\widetilde{S}^1_u, \widetilde{S}^2_v)$. Then

$$\widetilde{S}_t(u,v) = \mathbb{E}\big[\mathcal{C}(\widetilde{S}^1_u, \widetilde{S}^2_v)|\mathcal{F}_t\big].$$

3.6.2 Propagation of correlation

Contrary to the previous example, we now propose models from a forward point of view. More precisely, we suppose given a certain correlation structure at the initial time and we shall study the "dynamics" of the correlation. In particular, we discuss the martingale property of the conditional joint distribution.

The following example shows that any initial correlation structure induced by a copula function can be diffused as a martingale.

Example 3.15 For any $t, u, v \geq 0$, let

$$\widetilde{S}_t(u,v) = \exp\Big(- \big(u^2 M^1_t + v^2 M^2_t\big)^{1/2} - A_t\Big) \tag{21}$$

where

$$A_t = \frac{1}{8}\int_0^t \frac{1 + X_s^{1/2}}{X_s^{2/3}} d\left\langle X \right\rangle_s \quad \text{and} \quad X_s = u^2 M^1_s + v_2 M^2_s$$

with M^1 and M^2 being $\mathbb{F}$-martingales such that $\left\langle M^1, M^2\right\rangle_t > 0$.
It's not difficult to verify that (21) defines a conditional joint probability. In fact, $\widetilde{S}_t(u,v)$ is smaller than 1 and tends to 0 when u or v tends to infinity. In addition, it is decreasing with respect to u and v and the probability density is positive, i.e. $\partial_u \partial_v \widetilde{S}_t(u,v) \geq 0$. When $t = 0$, $\widetilde{S}_0(u,v) = \exp(-(u^2 + v^2)^{1/2})$, which is a cumulative distribution function of two exponential random variables with unit parameter linked with a Clayton copula.

4 Successive defaults

The aim of this section is twofold. We first provide generalization of the density framework to n successive default times. We then show how to apply these results to credit portfolio products and discuss the dynamic pricing of k^{th}-to-default swaps and CDOs.

4.1 Notation

We consider a family of default times $\tau_1, \cdots, \tau_n$ whose ordered set is denoted by $\sigma_1 \leq \sigma_2 \cdots \leq \sigma_n$. For any $i = 1, \cdots, n$, denote by $\mathbb{D}^i$ the filtration associated with τ_i and by $\mathbb{D}^{(i)}$ the filtration associated with σ_i. Let $\mathbb{F}$ be the reference filtration and define $\mathbb{G}$ as $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \cdots \vee \mathbb{D}^n$. In addition, let $\mathbb{D}^{(1,\cdots,i)} = \mathbb{D}^{(1)} \vee \cdots \vee \mathbb{D}^{(i)}$ and $\mathbb{G}^{(i)} = \mathbb{F} \vee \mathbb{D}^{(1,\cdots,i)}$. The fundamental hypothesis is the existence of the $\mathbb{F}$ density of $(\sigma_1, \cdots, \sigma_n)$.

Hypothesis 4.1 Assume that the conditional distribution of $(\sigma_1, \cdots, \sigma_n)$ given $\mathcal{F}_t$ admits a density w.r.t. the Lebesgue measure, i.e. for any $t \geq 0$ and $0 \leq \theta_1 \cdots \leq \theta_n$,

$$S_t(\theta_1, \cdots, \theta_n) := \mathbb{P}(\sigma_1 > \theta_1, \cdots, \sigma_n > \theta_n | \mathcal{F}_t) = \int_{\theta_1}^{\infty} \cdots \int_{\theta_n}^{\infty} \alpha_t(u_1, \cdots, u_n) du_1 \cdots du_n. \quad (22)$$

It can be noted that α is null outside the set $\{\theta_1 \leq \cdots \leq \theta_n\}$.

Similar as in the two-names case, the marginal density of σ_i can be deduced under Hypothesis 4.1. In the following, we shall denote the $\mathbb{G}^{(i-1)}$ survival probability of σ_i by $S_t^i(\theta) := \mathbb{P}(\sigma_i > \theta | \mathcal{G}_t^{(i-1)})$ for any positive t and θ, and we use the notation $\alpha_t^i(\theta)$ to represent the $\mathbb{G}^{(i-1)}$ density of σ_i, i.e. $S_t^i(\theta) = \int_\theta^\infty \alpha_t^i(u) du$.

4.2 Generalizations

We have shown in Section 3 how to apply the before and after default analysis to two default times. We now generalize the previous results to n successive defaults. Using the recursive procedure, the formulas can be expressed in terms of processes and conditional expectations w.r.t. the reference filtration $\mathbb{F}$. For the convenience of writing, we introduce the following notation: for any $T, t \geq 0$, let

$$\alpha_{T,t}(Y_T) := \int_t^\infty \cdots \int_t^\infty Y_T(u_1, \cdots, u_n) \alpha_T(u_1, \cdots, u_n) du_1 \cdots du_n; \quad (23)$$

for any $i = 1, \cdots, n-1$,

$$\alpha_{T,t}^{(u_1,\cdots,u_i)}(Y_T) := \int_t^\infty \cdots \int_t^\infty Y_T(u_1, \cdots, u_n) \alpha_T(u_1, \cdots, u_n) du_{i+1} \cdots du_n; \quad (24)$$

and

$$\alpha_{t,t}^{(u_1,\cdots,u_n)}(Y_t) := Y_t(u_1, \cdots, u_n) \alpha_t(u_1, \cdots, u_n). \quad (25)$$

For n successive defaults, there exist $n+1$ default scenarios according to the number of defaults. The following result is a generalization of Theorem 3.8 and computes the $\mathbb{G}^{(n)}$-conditional expectations.

Proposition 4.2 *Let $Y_T(\sigma_1,\cdots,\sigma_n)$ be a random variable such that $(\omega,T;u_1,\cdots,u_n)\to Y_T(u_1,\cdots,u_n;\omega)$ is $\mathcal{F}_T\times\mathcal{B}(\mathbb{R}_+^n)$-measurable. Then*

$$\mathbb{E}\big[Y_T(\sigma_1,\cdots,\sigma_n)|\mathcal{G}_t^{(n)}\big]=\sum_{i=0}^{n}1_{\{\sigma_i\le t<\sigma_{i+1}\}}\,q_t^i(T,Y_T) \tag{26}$$

where

$$q_t^i(T,Y_T)=\frac{\mathbb{E}[\alpha_{T,t}^{(u_1,\cdots,u_i)}(Y_T)\,|\,\mathcal{F}_t]}{\alpha_{t,t}^{(u_1,\cdots,u_i)}(1)}\Big|_{\substack{u_1=\sigma_1\\ \cdots\\ u_i=\sigma_i}}. \tag{27}$$

We use the convention $\sigma_0=0$ and $\sigma_{n+1}=\infty$.

Proof. Any $\mathcal{G}_t^{(n)}$-measurable random variable Z can be written as $F(t,\sigma_1\wedge t,\dots,\sigma_n\wedge t)$. Equality (26) is equivalent to

$$\mathbb{E}\big[Y_T(\sigma_1,\dots,\sigma_n)Z\big]=\sum_{i=0}^{n}\mathbb{E}(Z1_{\{\sigma_i\le t<\sigma_{i+1}\}}\,q_t^i(T,Y_T))$$

for any Y of the form $X_Tg(\sigma_1,\dots,\sigma_n)$ where X_T is $\mathcal{F}_T$-measurable and g is a Borel function. Note that $1_{\{\sigma_i\le t<\sigma_{i+1}\}}Z=F(t,\sigma_1,\dots,\sigma_i,t,\dots,t)$. Then we verify that (26) holds. □

Proposition 4.2 provides an explicit form of $\mathbb{G}^{(n)}$-martingales which are given as conditional expectation of $\mathcal{G}_T^{(n)}$ random variable. We now study the characterization of $\mathbb{G}^{(n)}$-martingales in terms of $\mathbb{F}$-martingales. Any $\mathbb{G}^{(n)}$-martingale $Y^{(n)}$ can be decomposed into the sum of $(n+1)$ processes $Y^{i,(n)}, i\in\{0,\cdots,n\}$ where $Y^{i,(n)}$ is a $\mathbb{G}^{(n)}$-adapted process starting at the i^{th} default σ_i and stopped at the $(i+1)^{\text{th}}$ default σ_{i+1}. We still use convention $\sigma_0=0$ and $\sigma_{n+1}=\infty$. For this reason, we shall concentrate on processes which are of the form

$$Y_t^{i,(n)}=1_{\{\sigma_i\le t<\sigma_{i+1}\}}Y_t(\sigma_1,\cdots,\sigma_i)+1_{\{\sigma_{k+1}\le t\}}\overline{Y}_{\sigma_{i+1}}(\sigma_1,\cdots,\sigma_{i+1}). \tag{28}$$

The marginal $\mathbb{F}$-density of the first i defaults will be useful, for which we denote by, for any $t\ge0$ and any $0\le\theta_1\cdots\le\theta_i$,

$$\begin{aligned}\alpha_t^{i,(n)}(\theta_1,\cdots,\theta_i)d\theta_1\dots d\theta_i &:= \mathbb{P}(\sigma_1\in d\theta_1,\cdots,\sigma_i\in d\theta_i|\mathcal{F}_t)\\ &= d\theta_1\dots d\theta_i\int_{\theta_i}^{\infty}\cdots\int_{\theta_{n-1}}^{\infty}\alpha_t(\theta_1,\cdots,\theta_i,u_{i+1},\dots,u_n)du_{i+1}\dots du_n.\end{aligned}$$

The following result gives the martingale characterization for the decomposed process $Y^{i,(n)}$.

Proposition 4.3 *Let $Y_t(\sigma_1,\cdots,\sigma_i)$ and $\overline{Y}_t(\sigma_1,\cdots,\sigma_{i+1})$, $i\in\{0,\cdots,n\}$ be random variables such that $(\omega,t;u_1,\cdots,u_i)\to Y_t(u_1,\cdots,u_i)$ and $(\omega,t;u_1,\cdots,u_{i+1})\to\overline{Y}_t(u_1,\cdots,u_{i+1})$ are $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}_+^i)$ and $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}_+^{i+1})$ measurable respectively and define $Y^{i,(n)}$ as in (28). Then*

1) $(1_{\{\sigma_i \leqslant t\}} Y_t(\sigma_1, \cdots, \sigma_i), t \geq 0)$ *is a* $\mathbb{G}^{(i)}$*-martingale if and only if, for all* $0 \leqslant \theta_1 \leq \cdots \leq \theta_i$*, the process* $(Y_t(\theta_1, \cdots, \theta_i)\alpha_t^{i,(n)}(\theta_1, \cdots, \theta_i), t \geq \theta_i)$ *is an* $\mathbb{F}$*-martingale .*
2) $Y^{i,(n)}$ *is a* $\mathbb{G}^{(i+1)}$*-martingale if and only if*

$$\big(Y_t(\theta_1, \cdots, \theta_i)\alpha_{t,t}^{(\theta_1,\cdots,\theta_i)}(1) + \int_{\theta_i}^t \overline{Y}_s(\theta_1, \cdots, \theta_i, s)\alpha_s^{i+1,(n)}(\theta_1, \cdots, \theta_i, s)ds,\ t \geq 0\big)$$

is an $\mathbb{F}$*-local martingale where* $\alpha_{t,t}^{(\theta_1,\cdots,\theta_i)}$ *is defined in* (23)*,* (24) *and* (25)*.*

Proof. 1) We prove by induction on i. By the martingale characterization for the after default case (see 2) of Theorem 2.3), the assertion holds for $i = 1$. Suppose we have proved for $i-1$. Again by Theorem 2.3 applied to σ_i, we know that $(1_{\{\sigma_i \leq t\}} Y_t(\sigma_1, \cdots, \sigma_i), t \geq 0)$ is a $\mathbb{G}^{(i)}$-martingale if and only if $(1_{\{\sigma_{i-1} \leq t\}} Y_t(\sigma_1, \cdots, \sigma_{i-1}, \theta)\alpha_t^i(\theta), t \geq \theta)$ is a $\mathbb{G}^{(i-1)}$-martingale where we recall that $\alpha_t^i(\theta)$ is the $\mathbb{G}^{(i-1)}$-density of σ_i and is given by

$$1_{\{\sigma_{i-1} \leq t\}}\alpha_t^i(\theta) = 1_{\{\sigma_{i-1} \leq t\}} \frac{\alpha_t^{i,(n)}(\sigma_1, \cdots, \sigma_{i-1}, \theta)}{\alpha_t^{i-1,(n)}(\sigma_1, \cdots, \sigma_{i-1})}.$$

By the hypothesis of induction, we know that the above condition is equivalent to $(Y_t(\theta_1, \cdots, \theta_i)\alpha_t^{i,(n)}(\theta_1, \cdots, \theta_i), t \geq \theta_i)$ is an $\mathbb{F}$-martingale.
2) We still prove by induction. By 1) of Theorem 2.3, the assertion is true for $i = 0$. Apply then the martingale characterization result to σ_{i+1} and we obtain the condition

$$\big(1_{\{\sigma_i < t \leq \sigma_{i+1}\}} Y_t(\sigma_1, \cdots, \sigma_i)S_t^{i+1} + \int_0^t 1_{\{\sigma_i < s\}} \overline{Y}_s(\sigma_1, \cdots, \sigma_i, s)\alpha_s^{i+1}(s)ds, t \geq 0\big) \tag{29}$$

is a $\mathbb{G}^{(i)}$-local martingale. Since the conditional survival probability of σ_{i+1} given $\mathcal{G}_t^{(i)}$ and its density satisfy

$$1_{\{\sigma_i \leq t\}} S_t^{i+1} = 1_{\{\sigma_i \leq t\}} \frac{\int_t^\infty \alpha_t^{i+1,(n)}(\sigma_1, \cdots, \sigma_i, v)dv}{\alpha_t^{i,(n)}(\sigma_1, \cdots, \sigma_i)}$$

and

$$1_{\{\sigma_i \leq t\}}\alpha_t^{i+1}(t) = 1_{\{\sigma_i \leq t\}} \frac{\alpha_t^{i+1,(n)}(\sigma_1, \cdots, \sigma_i, t)}{\alpha_t^{i,(n)}(\sigma_1, \cdots, \sigma_i)},$$

we apply 1) to the $\mathbb{G}^{(i)}$-local martingale (29) starting at σ_i and obtain that the above condition is equivalent to

$$\Big(Y_t(\theta_1, \cdots, \theta_i)\alpha_{t,t}^{(\theta_1,\cdots,\theta_i)} + \alpha_t^{i,(n)}(\theta_1, \cdots, \theta_i)\int_0^t \overline{Y}_s(\theta_1, \cdots, \theta_i, s)\frac{\alpha_s^{i+1,(n)}(\theta_1, \cdots, \theta_i, s)}{\alpha_s^{i,(n)}(\theta_1, \cdots, \theta_i)}ds, t \geq 0\Big)$$

is an $\mathbb{F}$-martingale. Finally, by using the Itô formula, we obtain 2).

□

We now give the intensity of σ_i as an extension of Proposition 3.5. Note that the $\mathbb{G}^{(i)}$-intensity of σ_i coincides with those w.r.t. $\mathbb{G}^{(n)}$ and $\mathbb{G}$.

Proposition 4.4 *The $\mathbb{G}^{(i)}$-intensity process of σ_i is given by*

$$\lambda_t^{\sigma_i,\mathbb{G}^{(i)}} = 1_{\{\sigma_{i-1}<t\le\sigma_i\}} \frac{\alpha_{t,t}^{(\sigma_1,\cdots,\sigma_{i-1},t)}(1)}{\alpha_{t,t}^{(\sigma_1,\cdots,\sigma_{i-1})}(1)}$$

where the notation is as in (23), (24) *and* (25).

Proof. By Proposition 2.2,

$$\lambda_t^{\sigma_i,\mathbb{G}^{(i)}} = 1_{\{\sigma_i>t\}} \frac{\alpha_t^i(t)}{S_t^i(t)}.$$

As in Proposition 3.5, we have $\alpha_t^{(i)} = 0$ for any $t < \sigma_{i-1}$. So we only need to calculate on the set $\{\sigma_{i-1} \le t < \sigma_i\}$. The explicit result is based on Proposition 4.2 and we have

$$\lambda_t^{\sigma_i,\mathbb{G}^{(i)}} = 1_{\{\sigma_{i-1}\le t<\sigma_i\}} \frac{\int_t^\infty \cdots \int_{u_{n-1}}^\infty \alpha_t(\sigma_1,\cdots,\sigma_{i-1},t,u_{i+1},\cdots,u_n)du_{i+1}\cdots du_n}{\int_t^\infty \cdots , \int_{u_{n-1}}^\infty \alpha_t(\sigma_1,\cdots,\sigma_{i-1},u_i,\cdots,u_n)du_i\cdots du_n}.$$

It remains to use the notations introduced in (23), (24) and (25). □

4.3 Applications to credit portfolio derivatives

A credit derivative product is described in a general way by (X, G, Z, τ) (see [2]) where X is an $\mathcal{F}_T$-measurable random variable representing the payment at maturity T, G is an $\mathbb{F}$-adapted, continuous process of finite variation with $G_0 = 0$ and represents the dividend payment, Z is an $\mathbb{F}$-predictable process and represents the recovery payment at the default time τ. We denote the saving account by $B_t = \exp(\int_0^t r_u du)$. Then the value process of such a product is given by

$$V_t = B_t\, \mathbb{E}_{\mathbb{Q}}\Big[\int_{]t,T]} B_u^{-1} d\Delta_u \,|\, \mathcal{G}_t\Big] \tag{30}$$

where $\mathbb{Q}$ is some risk-neutral probability and Δ is the dividend process defined by

$$\Delta_t = X 1_{\{\tau>T\}} 1_{\{T\}}(t) + \int_{]0,t\wedge T]} (1-D_u)dG_u + \int_{]0,t\wedge T]} Z_u dD_u$$

with $D_t = 1_{\{\tau\le t\}}$ and $\mathcal{G}_t$ represents the accessible market information at time t.

For credit portfolio derivatives such as k^{th}-to-default swap and CDOs, the payoff functions only concern the ordered sets of n underlying credit names instead of individual names themselves. Hence the filtration $\mathbb{G}^{(n)}$ represents the market information.

4.3.1 Pricing of k^{th}-to-default swap

A k^{th}-to-default (ktd) swap provides to its buyer the protection against the k^{th} default of the underlying pool. The protection buyer pays a prefixed regular premium κ until the moment of the k^{th} default σ_k, or until the maturity T if there are less than k defaults before T. In return, the protection seller pays the loss of the buyer $1-R^{(k)}$ at time σ_k where $R^{(k)}$ is the recovery rate of the k^{th} default and is an $\mathbb{F}$-predictable process. The dividend process of a ktd swap is given by

$$\Delta_t = \int_{]0,t\wedge T]} (1-D_u)dG_u + \int_{]0,t\wedge T]} Z_u dD_u$$

where $D_t = 1_{\{\sigma_k \le t\}}$, $G_t = -\kappa t$ and $Z_t = 1 - R_t^{(k)}$. We are interested in its value process

$$V_t = B_t\, \mathbb{E}_{\mathbb{Q}}\Big[\int_{]t,T]} B_u^{-1} d\Delta_u \,|\, \mathcal{G}_t^{(n)}\Big].$$

Proposition 4.5 *The value process of a ktd swap is given by*

$$V_t = B_t \sum_{i=0}^{k-1} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} q_t^i\Big(T, \int_t^T B_u^{-1} S_u^{k,\mathbb{F}} \kappa du - \int_t^T B_u^{-1} Z_u \alpha_u^{k,\mathbb{F}}(u) du\Big)$$

where q_t^i is defined by (27) under the risk-neutral probability $\mathbb{Q}$, $S^{k,\mathbb{F}}$ is the marginal $\mathbb{F}$ survival process of σ_k, i.e. $S_t^{k,\mathbb{F}} = \mathbb{P}(\sigma_k > t|\mathcal{F}_t)$ and $\alpha^{k,\mathbb{F}}$ is the marginal $\mathbb{F}$ density of σ_k. By convention, $\sigma_0 = 0$ and $\sigma_{n+1} = \infty$.

Proof. By definition, we have

$$V_t = B_t\, \mathbb{E}_{\mathbb{Q}}\Big[\int_{]t,T]} B_u^{-1} 1_{\{\sigma_k \ge u\}} \kappa du + B_{\sigma_k}^{-1} Z_{\sigma_k} \,|\, \mathcal{G}_t^{(n)}\Big].$$

Let $U_s = \int_t^s B_u^{-1} \kappa du$ and $L_s = U_s + B_s^{-1} Z_s$, then

$$V_t = B_t\, \mathbb{E}_{\mathbb{Q}}\big[1_{\{\sigma_k > T\}} U_T + 1_{\{t < \sigma_k \le T\}} L_{\sigma_k} \,|\, \mathcal{G}_t^{(n)}\big]. \tag{31}$$

The first term can be calculated directly by applying Proposition 4.2,

$$\mathbb{E}\big[1_{\{\sigma_k > T\}} U_T | \mathcal{G}_t^{(n)}\big] = \sum_{i=0}^{k-1} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} q_t^i(T, U_T S_T^{k,\mathbb{F}}).$$

For the second term, we have

$$\mathbb{E}[1_{\{t < \sigma_k \le T\}} L_{\sigma_k} | \mathcal{G}_t^{(n)}] = -\sum_{i=0}^{k-1} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} q_t^i\big(T, \int_t^T L_u dS_u^{k,\mathbb{F}}\big).$$

Combining the two terms and noticing that $U_t = 0$, we obtain

$$U_T S_T^{(k)} - \int_t^T L_u dS_u^{k,\mathbb{F}} = \int_t^T S_u^{k,\mathbb{F}} dU_u - \int_t^T B_u^{-1} Z_u dS_u^{k,\mathbb{F}}.$$

Finally, we remark that $(\alpha_t^{k,\mathbb{F}}(t), t \geq 0)$ is the density process of the increasing predictable process in the Doob-Meyer decomposition of the supermartingale $S^{k,\mathbb{F}}$, which implies the proposition.

□

4.3.2 Pricing of CDO tranches

The key term for pricing a CDO tranche is the cumulative loss $(l_t, t \geq 0)$ on the underlying portfolio. Under the simplified hypothesis that the notional value and the recovery rate of each credit name are identical, we may suppose $l_t = \sum_{i=1}^n 1_{\{\tau_i \leq t\}} = \sum_{i=1}^n 1_{\{\sigma_i \leq t\}}$, which takes integer values on the set $\{0, 1, \cdots, n\}$. In the successive defaults approach, $\{l_t = k\} = \{\sigma_k \leq t < \sigma_{k+1}\}$ for an integer $k = 0, 1, \cdots, n$. In addition, for any real number $u \leq n$, $\{l_t \leq u\} = \{\sigma_{\lfloor u \rfloor + 1} > T\}$ where $\lfloor u \rfloor$ is the integer part of u.

We now consider a CDO tranche with upper and lower barriers a and b. The value of the tranche at time $t < T$ depends on the loss on the tranche which is of form of a call spread $(l_t - a)^+ - (l_t - b)^+$. We are hence interested in computing $\mathbb{E}[(l_T - a)^+ - (l_T - b)^+ | \mathcal{G}_t^{(n)}]$ where $T > t$. By Proposition 4.2,

$$\begin{aligned}\mathbb{E}\big[(a - l_T)^+ | \mathcal{G}_t^{(n)}\big] &= \int_{-\infty}^a du\, \mathbb{E}\big[1_{\{l_T \leq u\}} | \mathcal{G}_t^{(n)}\big] = \int_{-\infty}^a du\, \mathbb{E}\big[1_{\{\sigma_{\lfloor u \rfloor + 1} > T\}} | \mathcal{G}_t^{(n)}\big] \\ &= \int_{-\infty}^a du \sum_{i=0}^{\lfloor u \rfloor} 1_{\{\sigma_i \leq t < \sigma_{i+1}\}} q_t^i\Big(T, S_T^{\lfloor u \rfloor, \mathbb{F}}\Big).\end{aligned}$$

Using the call-put parity, we hence deduce the value process of a CDO tranche.

5 Conclusion

In this paper, we have applied the density approach to successive default events. Under the reasonable hypotheses on the existence of joint density process with respect to the reference filtration $\mathbb{F}$, we have deduced dynamics of pricing processes for credit portfolio products.

The study is based on the before-default and after-default analysis and allows us to examine in detail the impact of one default event on the remaining credit names such as the contagious jump of the default intensity. Furthermore, the pricing formulas are given on different default scenarios and hence make clear the value change of a financial product due to the default event.

The dependence structures between default times are represented by the $\mathbb{F}$-joint density process and we have proposed several modelling methods. The key idea is to diffuse a static correlation structure at the initial time to achieve a "dynamic correlation". The density approach provides a new vision on the default dependence problems. Under this theoretical framework, some explicit models of joint density process may be studied in more detail for further practical use.

References

[1] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski, *Pricing and trading the credit default swaps in a hazard process model*, 2007.

[2] T.R. Bielecki and M. Rutkowski, *Credit risk : modeling, valuation and hedging*, Springer-Verlag, 2002.

[3] T.R. Bielecki, A. Vidozzi, and L. Vidozzi, *A markov copulae approach to pricing and hedging of credit index derivatives and ratings triggered step–up bonds*, J. of Credit Risk **4** (2008), no. 1.

[4] T.R. Bielecki, A. Vidozzi, L. Vidozzi, and J. Jakubowski, *Study of dependence for some stochastic processes*, Journal of Stochastic Analysis and Applications **26** (2008), no. 4, 903–924.

[5] D. Brigo, A. Pallavicini, and R. Torresetti, *Calibration of cdo tranches with the dynamical generalized-poisson loss model*, working paper, 2007.

[6] P. Ehlers and P. Schönbucher, *Background filtrations and canonical loss processes for top-down models of portfolio credit risk*, Finance and Stochastics **13(1)** (2009), 79–103.

[7] N. El Karoui, M. Jeanblanc, and Y. Jiao, *What happens after a default : the conditional density approach*, Working paper, 2008.

[8] R. Frey and A.J. McNeil, *Modelling dependent defaults*, Working paper, 2001.

[9] K. Giesecke, *The correlation-neutral measure for portfolio credit*, Working paper, Stanford University, 2007.

[10] A. Herbertsson, *Modelling default contagion using multivariate phase-type distributions*, Working paper, Göteborg University, 2007.

[11] P.J. Schönbucher, *Portfolio losses and the term structure of loss transition rates: A new methodology for the pricing of portfolio credit derivatives*, Working paper, 2006.

[12] P.J. Schönbucher and D. Schubert, *Copula-dependent default risk in intensity models*, Working paper, University of Bonn, 2001.