THE VAR AT RISK

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Abstract. I show that the structure of the firm is not neutral in respect to regulatory capital budgeted under rules which are based on the Value-at-Risk.

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Introduction. The single most used measure of financial risk is undoubtedly the Value-at-Risk (VaR). The VaR at level 95% is defined as the minimal amount of capital which is required to cover the losses in 95% of cases. In statistical terms, the value-at-risk is the quantile of level α of the losses, namely

$$VaR_{\alpha}(X) = \inf\{x : \Pr(X \leq x) > \alpha\}.$$

(note that unlike the most widely adopted convention in the literature, we chose to count positively an effective loss).

The widespread popularity of the Value-at-Risk is due to its adoption as a the "1st pillar" in the Basle II agreements. Despite its widespread use and simplicity, the Value-at-Risk is highly criticized among academics. The literature on risk measures classically defines a set of axioms which satisfactory risk measures should satisfy (see [1]). Among these, *subadditivity*: a risk measure ρ should satisfy $\rho(X+Y) \leq \rho(X) + \rho(Y)$. While this axiom is widely accepted in the academic community, it is perhaps ironical that VaR fails to satisfy this subadditivity axiom as it has been widely documented (we come back to that below).

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The subadditivity axiom is generally motivated by a loose invocation of risk aversion, or preference for diversification. In this note I propose a quite different argument to motivate the importance of the subadditivity axiom. Assuming the value-at-risk is used to budget regulatory capital requirements, and assuming that the managers have an incentive to minimize these capital requirements, I show that the lack of the subadditivity property induces the possibility for the management to optimally divide their risk in order to minimize their budgeted capital: the structure of the firm is not neutral to the aggregated capital requirement. More precisely, I show that for any level of the value-at-risk there is a division of the risk which sets the aggregated capital requirement to zero.

The problem. Consider a trading floor which is organized into N various desks. For each trading desk $i = 1, ..., N$, call X_i the random variable of the contingent loss of trading floor i. The total random loss of the trading floor is $X = \sum_{i=1}^{N} X_i$, we suppose that this amount is bounded: $X \in [0, M]$ almost surely.

We suppose that each desk budgets a regulatory capital equal to its VaR at level $\alpha \in (0, 1)$, $VaR_\alpha(X_i)$. Consequently the total amount of regulatory capital which the management needs to budget is $\sum_{i=1}^{N} VaR_\alpha(X_i)$. It is therefore easy to formulate the manager's problem:

$$\inf \sum_{i=1}^{N} VaR_\alpha(X_i) \text{ s.t. } \sum_{i=1}^{N} X_i = X \text{ a.s., and } VaR_\alpha(X_i) \geq 0.$$

As the management has full control over the structure of the firm, it results that it has the choice over N and over the random variables $(X_1, ..., X_N)$ which satisfies the constraints.

Note that the economical risk which the trading floor bears is $X = \sum_{i=1}^{N} X_i$, and thus the regulatory capital to be budgeted should be $VaR_\alpha(X) > 0$. However, I shall show that, under mild assumptions, there is a structure of the firm such that the capital budgeted under the rule above is zero. The assumption needed is the following:

Assumption. *The distribution of X is absolutely continuous with respect to the Lebesgue measure.*

The optimal structure. In that case, there exists a sequence of real numbers $x_0 = 0 < x_1 < ... < x_N = M$ such that $\Pr(X \in (x_i, x_{i+1})) < 1 - \alpha$ for all $i = 1, ..., N - 1$.

We are therefore going to consider the *digital options*[1]

$$X_i = X1\left\{X \in (x_i, x_{i+1})\right\}$$

(note that these options can be approximated by linear combinations of standard calls and puts, or *butterfly options*). We have

$$\sum_{i=1}^{N} X_i = X.$$

As we have $\Pr(X_i > 0) < 1 - \alpha$, it follows that

$$VaR_\alpha(X_i) = 0.$$

Therefore the capital to be budgeted under this structure of the firm is the sum of the capital to be budgeted for the different trading desks, which is zero.

Remarks.

1. This example shows that it is possible to tear down the risk into small pieces which are indetectable by the Value-at-Risk.

2. Note that the case we have here, which is the worse case (at least for the regulator's point of view), is the case where all the X_i's are *comonotonic*. Comonotonicity indeeds turns out to be the regulator's worse case, which justifies the *comonotonic additivity* axiom put forward a large literature: see [11].

3. The above considerations have lead [7] to define the axiom of *strong coherence*, which is a natural requirement so that the structure of the firm be neutral to risk measurement. In that paper, strong coherence is shown to be equivalent to the classical risk measures axioms.

4. The assumption that the distribution of the risk X be absolutely continuous is crucial. There are connections to be explored with the theory of portfolio diversification under thick-tailedness, see [9].

[1] this is a slight abuse of terminology as a digital option would more correctly characterize $x_i1\left\{X \in (x_i, x_{i+1})\right\}$, but both expressions are close when x_i and x_{i+1} are close.

Possible extension. It would be interesting to consider the above problem with overhead costs, namely when dividing the firm into several desks is costly. The problem would then become

$$\inf \sum_{i=1}^{N} VaR_{\alpha}\left(X_i\right) + \alpha\left(N\right) \text{ s.t. } \sum_{i=1}^{N} X_i = X \text{ a.s., and } VaR_{\alpha}\left(X_i\right) \geq 0.$$

where $\alpha\left(N\right)$ is increasing with N and can be interepreted as an *a priori penalization of the firm's complexity* by the regulator. The problem of determining an optimal $\alpha\left(N\right)$ from the point of view of the regulator in a properly defined setting is of interest.

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