# Modeling of CDO Options with multi-period Spread Dynamics

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June 2008

**Abstract**

This paper provides a multi-period extension of the Market Model for forward-start options written on standardized CDS Index Tranches, presented by [3]. Standardized CDS Index Tranches are tranches which securitize CDS index series and dispose of predefined subordination. The central idea consists in defining the forward Fair Tranche Spread as a function of the numeraire used in a Black & Scholes closed-form market formula. Hence it becomes possible to select any martingale dynamics for the forward spread rate under the associated probability and one can derive a multi-period dynamics of the forward Fair Tranche Spread for any forward-time horizons. This model is useful for pricing options on tranches with future Issue Dates as well as for modeling emerging options on credit derivatives which include structural enhancements like combo, step-up, or ratio indexed coupon clauses. It becomes also possible to calibrate Index Tranche Options with bespoke tenors/tranche subordination to market data obtained by more liquid Index Tranche Options with standard characteristics.

**EFM - Classification:** 310, 340, 410, 420

**JEL - Classification:** G12, G13

**Keywords:** CDO, CDS, Index Tranche, Market Model, Tranche Spread, Forward

# Introduction

As standardization for structured credit derivatives, notably for CDOs is becoming more and more popular, the volumes traded on the market have been growing in a proportional manner.
However initially CDOs were and still are OTC products by definition and thus generate high transaction costs. In addition to this the liquidity gap costs some precious basis points to investors. In order to meet these problems, investors set up a standardized CDO market several years ago. The standardization assumptions consist in the following:

- The underlying CDS portfolio is restricted to components of CDX or iTRAXX CDS index series quoted on the market.
- Tranches are defined by set Attachment / Detachment Points.

This market has experienced a tremendous success and thus became quite liquid. One can actually face the possibility of trading options on these standardized CDO tranches. For this purpose we propose in this article a Market Model inspired by Brigo & Mercurio's CDS spread Market Model (2006) in order to price this recent derivative asset class.

The authors recall basic principles of CDO pricing in the preliminary section and notably define the Protection Leg without the industry-standard approximation. The reader will discover that this has a significant impact for the value of senior tranches.
The second section re-defines in a first attempt the one-period forward Tranche Spread presented previously in [3]. However the authors derive the latter in a different manner which does not require the freezing of the forward interest rate at the instant time. It follows the calculation of the multi-period Tranche Spread dynamics which allows for the pricing of real-world deals, i.e. deals with a maturity of several years.

The advantages of this model consist in the ability to price deals with future issue dates. These structures are not martingales as the forward-start tenor interrupts filtration. It follows that calculating the expectation of the multi-period spread dynamics represents the only correct solution for pricing.
Further, calibration to similar deals with different maturities finally makes it possible to take into account credit spread (and hence implied risk) volatility. This is not the case with current static models and hence the investor does not pay for the vega!

# 1 Preliminaries

This section recalls determinant elements for tranche pricing and settles the notation used throughout the article.

## 1.1 General Notations

In the following table we define frequently used notations:

| | |
|---|---|
| $N$ | total number of underlying CDSs |
| $k$ | Counting Index related to underlying CDSs |
| $T_a$ | Starting Date of the CDO Option |
| $T_b$ | Maturity Date of the CDO Option |
| $\tau_k$ | Default Time of CE $k$ in underlying portfolio |
| $A\%$ | Attachment Point of CDO tranche |
| $B\%$ | Detachment Point of CDO tranche |
| $N_k$ | Notional of credit entity $k$ |
| $N_{[A;B]}$ | Notional or thickness of $CDO$ tranche $[A;B]$ |
| $s$ | Deal Spread |
| $S_{a,b}^{[A,B]}$ | Forward Fair Tranche Spread associated to tranche $[A;B]$ in the interval $[T_a;T_b]$ |
| $\sigma_{a,b}$ | Volatility of $S_{a,b}^{[A,B]}$ |
| $Q^t(\Gamma(t)=d)$ | forward-neutral Probability of having exactly $d$ defaults on CDO tranche [A;B] by time horizon $t$ |
| $F(x)$ | Cumulative Probability Loss Function |
| $B(t,T)$ | Zero-Coupon Bond paying one unit at $t=T$ |
| $A(t,T)$ | Discount Factor (discounts from $T$ to $t$) |
| $R$ | Constant Recovery Rate (in %) |

With the notations presented in the table above, we can describe the following setting: The default of credit entity $k$ causes an erosion of the $CDO$ notional of

$$(1-R)\times N_k$$

and one can write

$$N_{CDO} = \sum_{k=1}^{N}(1-R)\times N_k$$

$$N_{[A;B]} = (B-A)\%\times N_{CDO} = (B-A)\%\times\sum_{k=1}^{N}(1-R)\times N_k$$

$$F(x,t) = \sum_{k=0}^{x} Q^t(\Gamma(t)=k)$$

### 1.2 The CDO Premium-Leg

The investor in a synthetic CDO tranche sells protection on the underlying CDS portfolio as he bears default risk. As a compensation, he perceives a spread premium (risk-neutral interest rate + risk premium) which is calculated as a [%] value on the tranche notional. However, as mentioned before, we do have to consider default events. When the cumulative Portfolio Loss reaches the tranche's Attachment Point, the tranche's notional gets eroded with every further default on the CDS portfolio until it wipes out. Hence the investor's spread revenue (and thus the Premium Leg) has to be calculated on the *outstanding* tranche notional.
The Premium Leg is defined as the sum of the discounted spread payments perceived by the tranche holder during the tranche's lifetime.

Consider an investor who holds a $[A\%; B\%]$ tranche of a synthetic $CDO$. In our framework, the underlying CDS portfolio consists of 125 CDS names figuring in a CDS index (iTraXX or CDX, for example). As we treat only *standardized* index tranches, the tranche's Attachment/Detachment Points $(A\%, B\%)$ can only take the following values:

$$0\%, 3\%, 6\%, 9\%, 12\%, 22\%, 100\%$$

Further we choose the following notations:

$$A = A\% N_{CDO}$$
$$B = B\% N_{CDO}$$

This is relevant for calculations where leverage intervenes.
We assume that the risk-free interest rate $r$ is constant and introduce the random cumulative loss variable $Z(t)$ with

$$Z(t) = \sum_{k=0}^{N} (1-R) N_k 1_{\{\tau_k < t\}} \tag{1}$$

**Proposition 1.1.** *The outstanding tranche's notional at time horizon $t$ satisfies*

$$X(t) = [B - Z(t)]^+ - [A - Z(t)]^+ \tag{2}$$

*Proof.* The bounded tranche loss with lower bound $A$ and upper bound $B$ can be written as

$$\begin{aligned} X(t) &= (B-A)1_{\{Z(t)<A\}} + (B - Z(t))1_{\{A \le Z(t) < B\}} && (3)\\ &= (B-A)1_{\{Z(t)<A\}} + (B - Z(t))\left(1_{\{Z(t)<B\}} - 1_{\{Z(t)<A\}}\right) && (4)\\ &= (Z(t)-A)1_{\{Z(t)<A\}} + (B - Z(t))1_{\{Z(t)<B\}} && (5)\\ &= [B - Z(t)]^+ - [A - Z(t)]^+ && (6) \end{aligned}$$

□

**Definition 1.1.** *For time intervals (or coupon dates) $t = (t_1, \ldots, t_p)$ the Premium Leg amounts to*

$$\begin{aligned} \textit{Premium Leg} &= s \times \sum_{i=1}^{p} E_Q\left[A(0,t_i)X(t_i)\right]\delta_i && (7)\\ &= s \times \sum_{i=1}^{p} E_{Q^{t_i}}\left[X(t_i)\right]B(0,t_i)\delta_i && (8) \end{aligned}$$

*with*

$$\delta_i = t_i - t_{i-1} \forall i \in (1, \ldots, p)$$

*and $Q^{t_i}$ representing the $t_i$-forward neutral probability measure.*

In most cases however, the default time of a credit name is situated between two regular spread payment dates. In this case, the spread payment has to be accrued with respect to the elapsed time between the last regular payment date and the default time. The Premium Leg gets increased by the Accrual.

**Definition 1.2** (Accrual)**.** *The Accrual of the Premium Leg is calculated as follows:*

$$\textit{Accrual(0)} = A(0,t_i) \times s \times \int_{t_i}^{\tau} \Delta u \times dE_Q[A(t_i, t_i + \Delta u)X(u)] \tag{9}$$

*with $\Delta t$ representing the accrued time.*

For simplification we neglect in the following the Accrual term and evaluate the Premium Leg like in the case of a postponed CDS.

**Let us focus on the expression $E_{Q^{t_i}}[X(t_i)]$:** We are interested in how the total losses affect the $CDO$ tranche $[A\%; B\%]$. From the payment subordination we know that the tranche suffers a loss at time $t$ if:

$$A\%N_{CDO} < Z(t) < B\%N_{CDO}$$

**When considering the tranche's erosion default by default, one obtains a discrete closed form expression:**

**Proposition 1.2.** *The Expected outstanding percentage Tranche Notional at time horizon $t$ can be written as*

$$E_{Q^t}\left[\hat{X}(t)\right] = (B\% - A\%)\sum_{k=0}^{\hat{x}} Q^t\left(\Gamma(t) = k\right) + B\% \times \sum_{k=\hat{x}+1}^{\hat{y}} Q^t\left(\Gamma(t) = k\right) - \tag{10}$$

$$- \sum_{k=\hat{x}+1}^{\hat{y}} \frac{N_k}{N_{CDO}} \times k \times Q^t\left(\Gamma(t) = k\right) \tag{11}$$

*with*

$$\hat{x} = Int\left[\frac{A\%N_{CDO}}{N_k}\right]$$

$$\hat{y} = Int\left[\frac{B\%N_{CDO}}{N_k}\right]$$

*Proof.*

$$\hat{X}(t) = \left[B\% - \hat{Z}(t)\right]^{+} - \left[A\% - \hat{Z}(t)\right]^{+} \tag{12}$$

$$= (B\% - A\%)1_{\{\hat{Z}(t) \le A\%\}} + (B\% - \hat{Z}(t))1_{\{A\% \le \hat{Z}(t) \le B\%\}} \tag{13}$$

When $k$ denotes the number of credit events on the underlying portfolio, $l_k$ stands for the corresponding [%] portfolio erosion and we define

$$l_k = \frac{k \times N_k}{N_{CDO}}$$

Thus

$$E_{Q^t}\left[\hat{X}(t)\right] = (B\% - A\%)Q^t\left(\hat{Z}(t) < A\%\right) + B\% \times Q^t\left(A\% \leq \hat{Z}(t) < B\%\right) \quad (14)$$

$$- E_{Q^t}\left[\hat{Z}(t)1_{\{A\%\leq\hat{Z}(t)<B\%\}}\right] \quad (15)$$

$$= (B\% - A\%)\sum_{l_k\in[0\%;A\%[} Q^t(\hat{Z}(t) = l_k) + B\% \times \sum_{l_k\in[A\%;B\%[} Q^t(\hat{Z}(t) = l_k) \quad (16)$$

$$- \sum_{l_k\in[A\%;B\%[} l_k \times Q^t(\hat{Z}(t) = l_k) \quad (17)$$

$$= (B\% - A\%)\sum_{k=0}^{\hat{x}} Q^t(\Gamma(t) = k) + B\% \times \sum_{k=\hat{x}+1}^{\hat{y}} Q^t(\Gamma(t) = k) \quad (18)$$

$$- \sum_{k=\hat{x}+1}^{\hat{y}} \frac{N_k}{N_{CDO}} \times k \times Q^t(\Gamma(t) = k) \quad (19)$$

with

$$k = l_k \times \frac{N_{CDO}}{N_k}$$
$$\hat{Z}(t) = \frac{Z(t)}{N_{CDO}}$$

□

Note:

$$E_{Q^t}\left[[Z(t) - A]^+ - [Z(t) - B]^+\right]$$

is called the Expected Tranche Loss and $Q^t(\Gamma(t) = k)$ represents the probability of having *exactly* $k$ default events on the underlying portfolio by time horizon $t$. The reader who is interested in how this probability is obtained, should refer to the Appendix in [3]. An explanation at this stage would be too exhaustive and confusing.

**When it comes to pricing any tranche throughout the deal's notional, one has to come up with a continuous expression:** With

$$l_k = \frac{k \times N_k}{N_{CDO}}$$

being the [%] loss caused by exactly $k$ defaults on the $CDO$ portfolio, we can write:

$$Q^t\left(\Gamma(t) = k\right) = Q^t\left(\Gamma(t) = l_k\right) \quad (20)$$

as the probability of having $k$ defaults on the $CDO$ portfolio is the same as having a $l_k\%$ erosion of the $CDO$ portfolio caused by $k$ defaults.
By analogy with

$$F(x) = \sum_{k=0}^{x} Q^t\left(\Gamma(t) = k\right)$$

we can write

$$F(l_x) = \int_0^{l_x} Q^t\left(\Gamma(t) = l_k\right) dl_k$$

with $l_x$ being the tranche-[%] equivalent to a $x$ default portfolio erosion.
Further let

$$f(l_x, t) = \frac{\partial F(l_x)}{\partial l_x}$$

be the density.

**Proposition 1.3.** *The Expected outstanding percentage CDO Tranche Notional can be written as:*

$$E\left[\hat{X}(t)\right] = [B\% - A\%] \int_0^{A\%} f(l_x, t) dl_x + B\% \times \int_{A\%+}^{B\%} f(l_x, t) dl_x - \tag{21}$$

$$- \int_{A\%+}^{B\%} l_x f(l_x, t) dl_x \tag{22}$$

*Proof.* With respect to the tranche's boundaries one can state

$$E_{Q^t}\left[\hat{X}(t)\right] = (B\% - A\%) Q^t\left(\hat{Z}(t) < A\%\right) + B\% \times Q^t\left(A\% \leq \hat{Z}(t) < B\%\right) \tag{23}$$

$$- E_{Q^t}\left[\hat{Z}(t) 1_{\{A\% \leq \hat{Z}(t) < B\%\}}\right] \tag{24}$$

$$= (B\% - A\%) Q^t(\Gamma(t) \leq \hat{x}) + B\% \times Q^t(\hat{x} < \Gamma(t) \leq \hat{y}) \tag{25}$$

$$- \sum_{l_k \in [A\%; B\%[} l_k \times Q^t(\hat{Z}(t) = l_k) \tag{26}$$

which amounts to 1.3. □

## 1.3 The CDO Protection Leg

The investor in a synthetic CDO tranche sells protection on the underlying CDS portfolio. When cumulative portfolio loss reaches the Attachment Point of his tranche, the tranche's notional gets eroded by every further default. As the tranche holder perceives a [%] spread on the outstanding tranche's notional, his earnings diminish with every further default. In a way, the investor "pays" for the default event.
That reduction of the tranche's notional inherent to a default on the underlying CDS portfolio is interpreted as a protection payment.
The Protection Leg is defined as the sum of the discounted reductions of a tranche's notional inherent to credit events which lead to a decrease in the tranche holder's "spread revenue".

**Definition 1.3.** *For time intervals $t = (t_1, \ldots, t_p)$ the Protection Leg is given by:*

$$Prot\ Leg = \sum_{i=1}^{p} \left(E_Q\left[A(0, t_{i-1}) X(t_{i-1})\right] - E_Q\left[A(0, t_i) X(t_i)\right]\right) \tag{27}$$

$$= \sum_{i=1}^{p} \left(B(0, t_{i-1}) E_{Q^{t_{i-1}}}\left[X(t_{i-1})\right] - B(0, t_i) E_{Q^{t_i}}\left[X(t_i)\right]\right) \tag{28}$$

$$\approx \sum_{i=1}^{p} B(0, t_i) \left(E_{Q^{t_i}}\left[X(t_{i-1})\right] - E_{Q^{t_i}}\left[X(t_i)\right]\right) \tag{29}$$

A continuous formula can be derived by applying the continuous time expression for the outstanding tranche notional as derived previously.

## 1.4 Defining a forward Fair Tranche Spread

When a *CDO* deal is set up, the coupon payment for the tranche holder is fixed in the deal's term sheet. Generally the coupon payment is defined as follows

$$Euribor/Libor(3months) + s$$

where $s$ stands for the *Deal Spread*, i.e. the spread which in addition to the Euribor/Libor Payment equals the Premium Leg and the Protection Leg at Issue Date.

However as time goes by, the presumed default pattern at issue date has changed and this is exactly why the tranche does not quote at 100% any more (The Default Leg has a different value than the Protection Leg). In order to quote a CDO tranche like a bond (i.e. in %) it is quite common to quote this tranche by its *Fair Spread*. The *Fair Spread* is the spread that should have been paid starting from the issue date instead of the pre-determined Deal Spread + Euribor/Libor such that the tranche would quote at 100% today (and hence sets the tranche's swap value to 0). One can write

$$\text{Fair Spread} = \frac{\text{Protection Leg}}{\text{Premium Leg}}$$

**Definition 1.4.** *The Forward Fair Tranche Spread paid on Tranche* $[A\%; B\%]$ *within the time interval* $[T_a; T_b]$ *is defined as follows:*

$$S_{a,b}^{A,B}(t) = \frac{\sum_{i=a+1}^{b} \left[B(t, T_{i-1}) E_{Q^{T_{i-1}}} \left[X(T_{i-1})\right] - B(t, T_i) E_{Q^{T_i}} \left[X(T_i)\right]\right]}{\sum_{i=a+1}^{b} \delta_i B(t, T_i) E_{Q^{T_i}} \left[X(T_i)\right]} \quad (30)$$

By the way $E_{Q^{T_i}}\left[X(T_i)\right]$ never sets to zero as the following reasoning illustrates: Any tranche never gets eroded continuously. Every default on the tranche causes a downward jump of the outstanding tranche notional. The jump size is $N_k$. Thus one can state:

$$E_{Q^T}\left[X(T)\right] = \sum_{k=0}^{N} Max\left(B - A - k \times N_k; 0\right) Q^T(\Gamma(T) = k) \quad (31)$$

As the default probability in an intensity-based default model is exponentially distributed, the default probability of any default scenario is never zero. Hence the expected outstanding tranche notional never equals zero.

## 2 The Model

In this section, the central idea consists in defining the forward Fair Tranche Spread in function of the numeraire used in a closed form Black & Scholes formula. Hence it becomes possible to select any martingale dynamics for the forward spread rate under the associated probability. Then we apply this approach for defining one-period forward Tranche Spread rates. Now as a last step we derive multi-period spread dynamics in order to replicate "real world" CDO deals.

We suppose the risk-free discount rate $r_t$ deterministic. Thus we can express the forward one-period spread associated to tranche $[A, B]$ with tenor $[T_a, T_b]$ as follows:

$$S_{a,b}^{A,B}(t) = \frac{\sum_{i=a+1}^{b} A(t,T_{i-1}) E_{Q^{T_{i-1}}}^{t}\left[X(T_{i-1})\right] - A(t,T_i) E_{Q^{T_i}}^{t}\left[X(T_i)\right]}{\sum_{i=a+1}^{b} \delta_i A(t,T_i) E_{Q^{T_i}}^{t}\left[X(T_i)\right]} \tag{32}$$

with

$$A(t,T_i) = \exp\left(-\int_t^{T_i} r_s ds\right) \tag{33}$$

and $Q$ stands for the risk-neutral probability measure.
Note that the fair spread's denominator never vanishes when applying the reduced form model: The default intensity is exponentially distributed, thus the expectation never equals zero. Consequently the spread is well defined.

$E_{Q^{T_i}}^{t}\left[X(T_i)\right]$ denotes the expected outstanding tranche notional at time horizon $T_i$ seen at $t$. This restricted information reflects the filtration $\mathcal{F}_t$. One can write

$$E_{Q^{T_i}}^{t}\left[X(T_i)\right] = E_{Q^{T_i}}\left[X(T_i)|\mathcal{F}_t\right]$$

For sake of simplification we note

$$Y_i(t) = E_{Q^{T_i}}^{t}\left[X(T_i)\right]$$

We consider the following numeraire

$$\hat{C}_{a,b}^{A,B}(t) = \sum_{i=a+1}^{b} \delta_i A(t,T_i) E_Q^t\left[X(T_i)\right] \tag{34}$$

$$= \sum_{i=a+1}^{b} \delta_i A(t,T_i) Y_i(t) \tag{35}$$

## 2.1 The One-Period Forward Spread Dynamics

In this section the aim consists in deriving the one-period forward spread dynamics associated to a CDO tranche $[A, B]$ with tenor $[T_{i-1}, T_i]$ in the general framework of an existing deal with tenor $[T_a, T_b]$. The calculations are in analogy to [3], however in this article we use no approximations in order to achieve the result.

**Proposition 2.1.** *The expected outstanding tranche notional $Y_i(t)$ is a $Q^t$-martingale. Its dynamics under the forward-neutral probability $Q^t$ follows :*

$$\frac{dY_i(t)}{Y_i(t)} = \gamma_i(t) dZ_t \tag{36}$$

*Proof.* In the one-period case, the forward Fair Spread for tranche $[A, B]$ with tenor $[T_{i-1}, T_i]$ can be written :

$$S^{A,B}_{i-1,i}(t) = \frac{A(t, T_{i-1}) E^t_{Q^{T_{i-1}}} [X(T_{i-1})] - A(t, T_i) E^t_{Q^{T_i}} [X(T_i)]}{\delta_i A(t, T_i) E^t_{Q^{T_i}} [X(T_i)]} \tag{37}$$

$$= \frac{A(t, T_{i-1}) Y_{i-1}(t) - A(t, T_i) Y_i(t)}{\delta_i A(t, T_i) Y_i(t)} \tag{38}$$

Hence one can consider the numéraire $\hat{C}^{A,B}_{i-1,i}(t) = \delta_i A(t, T_i) Y_i(t)$. We denote with $Q^{A,B}_{i-1,i}$ the probability associated to the numéraire $\hat{C}^{A,B}_{i-1,i}(t)$ and express the numeraire related to the risk-neutral probability as follows:

$$N^{Q^T}(T)\Big| \mathcal{F}_t = N^{Q^T,t}(T) = \exp\left(\int_t^T r_s ds\right) = \frac{1}{A(t, T)} \tag{39}$$

This amounts to the Radon-Nykodym derivative :

$$\left.\frac{d\hat{Q}^{A,B}_{i-1,i}}{dQ}\right|_{\mathcal{F}_t} = \frac{\hat{C}^{A,B}_{i-1,i}(T_{i-1}) N^{Q^t,t}(t)}{\hat{C}^{A,B}_{i-1,i}(t) N^{Q^{T_{i-1}},t}(T_{i-1})} \tag{40}$$

$$= \frac{\hat{C}^{A,B}_{i-1,i}(T_{i-1}) A(t, T_{i-1})}{\hat{C}^{A,B}_{i-1,i}(t)} \tag{41}$$

$$= \frac{\delta_i A(T_{i-1}, T_i) Y_i(T_{i-1}) A(t, T_{i-1})}{\delta_i A(t, T_i) Y_i(t)} \tag{42}$$

$$= \frac{Y_i(T_{i-1})}{Y_i(t)} \tag{43}$$

Hence $Y_i(t)$ is a $Q^t$-martingale. Its dynamics under $Q^t$ is given by :

$$\frac{dY_i(t)}{Y_i(t)} = \gamma_i(t) dZ_t \tag{44}$$

We consider $d$ forward spreads; for the one-period case the reader is invited to set $d = 1$. $\gamma_i$ represents the volatility vector $1 \times d$ of $Y_i$.Note that $Z_t$ is a column vector $d \times 1$ of correlated Brownian motions :

$$Z_t = \begin{pmatrix} Z_{1,t} \\ \vdots \\ Z_{d,t} \end{pmatrix}$$

Suppose $\rho$ being the correlation matrix $d \times d$ of one-period forward spreads related to the considered tenors, i.e. the tenor covers $d$ periods:

$$dZ_{i,t} dZ_{j,t} = \rho_{i,j} dt \tag{45}$$

Now we obtain the following Cholesky decomposition: $\rho = CC'$, with $C$ representing the lower triangular matrix. It follows under the risk-neutral probability $Q$:

$$Z_t = C \times W_t \tag{46}$$

with $W_t$ representing the column vector $d \times 1$ of standard Brownian motions under $Q$). The relationship between the different Brownian motions can be formalized as follows:

$$d\hat{Z}_t^{i-1,i} = dZ_t - \rho\left(\gamma_i(t)\right)' dt \tag{47}$$

where $(\gamma_i(t))'$ corresponds to the transposed volatility (line) vector of $Y_i$. □

Now that we have calculated the dynamics of the expected outstanding tranche notional at $T_i$ , i.e $Y_i$, the next step consists in determining its volatility.

$S_{i-1,i}^{A,B}(t)$ is a $Q_{i-1,i}^{A,B}$-martingale as its denominator is $\hat{C}_{i-1,i}^{A,B}(t)$. Consequently,

$$\frac{dS_{i-1,i}^{A,B}(t)}{S_{i-1,i}^{A,B}(t)} = \sigma_{i-1,i}(t) d\hat{Z}_t^{i-1,i} \tag{48}$$

In order to derive a dynamics we express the forward one-period spread in a simpler form:

$$S_{i-1,i}^{A,B}(t) = \frac{A(t,T_{i-1})Y_{i-1}(t) - A(t,T_i)Y_i(t)}{\delta_i A(t,T_i)Y_i(t)} \tag{49}$$

$$\delta_i S_{i-1,i}^{A,B}(t) = \frac{A(t,T_{i-1})Y_{i-1}(t)}{A(t,T_i)Y_i(t)} - 1 \tag{50}$$

$$1 + \delta_i S_{i-1,i}^{A,B}(t) = \frac{1}{A(T_{i-1},T_i)} \frac{Y_{i-1}(t)}{Y_i(t)} \tag{51}$$

We have to calculate the dynamics of $\frac{Y_{i-1}(t)}{Y_i(t)}$. With Ito's formula we have :

$$\frac{d\left(\frac{1}{Y_i(t)}\right)}{\frac{1}{Y_i(t)}} = \gamma_i(t)\rho\left(\gamma_i(t)\right)' dt - \gamma_i(t) dZ_t \tag{52}$$

The Leibniz rule gives :

$$d\left(\frac{Y_{i-1}(t)}{Y_i(t)}\right) = Y_{i-1}(t) d\left(\frac{1}{Y_i(t)}\right) + \frac{1}{Y_i(t)} d(Y_{i-1}(t)) + d(Y_{i-1}(t)) d\left(\frac{1}{Y_i(t)}\right) \tag{53}$$

$$= \frac{Y_{i-1}(t)}{Y_i(t)} \left[\gamma_i(t)\rho\left(\gamma_i(t)\right)' dt - \gamma_i(t) dZ_t + \gamma_{i-1}(t) dZ_t - \gamma_{i-1}(t)\rho\left(\gamma_i(t)\right)' dt\right] \tag{54}$$

$$= \frac{Y_{i-1}(t)}{Y_i(t)} \left[\left(\gamma_i(t) - \gamma_{i-1}(t)\right)\rho\left(\gamma_i(t)\right)' dt + \left(\gamma_{i-1}(t) - \gamma_i(t)\right) dZ_t\right] \tag{55}$$

$$= \frac{Y_{i-1}(t)}{Y_i(t)} \left(\gamma_{i-1}(t) - \gamma_i(t)\right)\left(dZ_t - \rho\left(\gamma_i(t)\right)' dt\right) \tag{56}$$

$$= \frac{Y_{i-1}(t)}{Y_i(t)} \left(\gamma_{i-1}(t) - \gamma_i(t)\right) d\hat{Z}_t^{i-1,i} \tag{57}$$

$$\tag{58}$$

Hence we obtain the following lemma:

**Lemma 2.2.** *$\forall k \in [a+1; b]$ the volatility of the process $Y_k$ related to tenor $[T_a, T_b]$ is given by*

$$\gamma_k(t) = -\sum_{j=a+1}^{k} \left( \frac{\delta_j S_{j-1,j}^{A,B}(t)}{1+\delta_j S_{j-1,j}^{A,B}(t)} \sigma_{j-1,j}(t) \right) \tag{59}$$

*Proof.* One can derive the dynamics of the forward one-period Fair Spread $S_{i-1,i}^{A,B}(t)$ by considering the differential of equation(51) :

$$d\left(1+\delta_i S_{i-1,i}^{A,B}(t)\right) = \delta_i dS_{i-1,i}^{A,B}(t) = \frac{1}{A(T_{i-1},T_i)} d\left(\frac{Y_{i-1}(t)}{Y_i(t)}\right) \tag{60}$$

$$= \frac{1}{A(T_{i-1},T_i)} \frac{Y_{i-1}(t)}{Y_i(t)} \left(\gamma_{i-1}(t) - \gamma_i(t)\right) d\hat{Z}_t^{i-1,i} \tag{61}$$

Hence

$$dS_{i-1,i}^{A,B}(t) = \frac{\left(1+\delta_i S_{i-1,i}^{A,B}(t)\right)}{\delta_i} \left(\gamma_{i-1}(t) - \gamma_i(t)\right) d\hat{Z}_t^{i-1,i} \tag{62}$$

We have seen previously that

$$\frac{dS_{i-1,i}^{A,B}(t)}{S_{i-1,i}^{A,B}(t)} = \sigma_{i-1,i}(t) d\hat{Z}_t^{i-1,i} \tag{63}$$

Consequently,

$$\gamma_{i-1}(t) - \gamma_i(t) = \frac{\delta_i S_{i-1,i}^{A,B}(t)}{1+\delta_i S_{i-1,i}^{A,B}(t)} \sigma_{i-1,i}(t) \tag{64}$$

In the general case with tenor $[T_a; T_b]$, we express $\gamma_k(t)$ for $k = a, \dots, b$. We showed that $Y_a(t) = E_{Q^{T_a}}^t [X(T_a)] = const.$, i.e $\gamma_a(t) = 0$. Thus $\forall k \in [a+1; b]$,

$$\gamma_k(t) = \sum_{j=a+1}^{k} \left(\gamma_j(t) - \gamma_{j-1}(t)\right) \tag{65}$$

$$= -\sum_{j=a+1}^{k} \left( \frac{\delta_j S_{j-1,j}^{A,B}(t)}{1+\delta_j S_{j-1,j}^{A,B}(t)} \sigma_{j-1,j}(t) \right) \tag{66}$$

□

At this state we are able to derive the dynamics of the forward one-period Fair Tranche Spreads which leads to the following lemma:

**Proposition 2.3.** *Consider a deal with tenor $[T_a, T_b]$ and tranche $[A, B]$. The dynamics of the forward one-period Fair Tranche Spread on tenor $[T_{i-1}, T_i]$ is given by:*

$$\frac{dS_{i-1,i}^{A,B}(t)}{S_{i-1,i}^{A,B}(t)} = \sigma_{i-1,i}(t)\rho \sum_{j=a+1}^{i} \left( \frac{\delta_j S_{j-1,j}^{A,B}(t)}{1 + \delta_j S_{j-1,j}^{A,B}(t)} \left(\sigma_{j-1,j}(t)\right)' \right) dt + \sigma_{i-1,i}(t) dZ_t \tag{67}$$

*More precisely, for a deal with tenor $[T_{i-1}, T_i]$, the forward one-period Fair Tranche Spread dynamics for the same tenor amounts to:*

$$\frac{dS_{i-1,i}^{A,B}(t)}{S_{i-1,i}^{A,B}(t)} = \frac{\delta_i S_{i-1,i}^{A,B}(t)}{1 + \delta_i S_{i-1,i}^{A,B}(t)} |\sigma_{i-1,i}(t)|^2 dt + \sigma_{i-1,i}(t) dW_t \tag{68}$$

*Proof.* With lemma (2.2), we express the forward one-period Fair Tranche Spread dynamics under the forward-neutral probability measure as follows :

$$\frac{dS_{i-1,i}^{A,B}(t)}{S_{i-1,i}^{A,B}(t)} = \sigma_{i-1,i}(t) d\hat{Z}_t^{i-1,i} \tag{69}$$

$$= \sigma_{i-1,i}(t) \left(dZ_t - \rho \left(\gamma_i(t)\right)' dt\right) \tag{70}$$

$$= \sigma_{i-1,i}(t) \left( dZ_t + \rho \left( \sum_{j=a+1}^{i} \left( \frac{\delta_j S_{j-1,j}^{A,B}(t)}{1 + \delta_j S_{j-1,j}^{A,B}(t)} \sigma_{j-1,j}(t) \right) \right)' dt \right) \tag{71}$$

$$= \sigma_{i-1,i}(t)\rho \sum_{j=a+1}^{i} \left( \frac{\delta_j S_{j-1,j}^{A,B}(t)}{1 + \delta_j S_{j-1,j}^{A,B}(t)} \left(\sigma_{j-1,j}(t)\right)' \right) dt + \sigma_{i-1,i}(t) dZ_t \tag{72}$$

The equation presented above apply for the general case. Thus for the specific one-period case for a deal with tenor $[T_{i-1}; T_i]$, we set $d = 1$. Consequently the correlation matrix shrinks to $\rho = 1$ and hence $dZ_t = dW_t$. Further more $\sigma_{i-1,i}$ becomes just a scalar. We obtain the following dynamics:

$$\frac{dS_{i-1,i}^{A,B}(t)}{S_{i-1,i}^{A,B}(t)} = \frac{\delta_i S_{i-1,i}^{A,B}(t)}{1 + \delta_i S_{i-1,i}^{A,B}(t)} |\sigma_{i-1,i}(t)|^2 dt + \sigma_{i-1,i}(t) dW_t \tag{73}$$

□

Now we have revealed the one-period spread dynamics, which has to be calibrated to liquid tenors, the aim consists in replicating a spread dynamics which covers the maturity of the deal structure to evaluate. For this purpose one has to develop the multi-period spread dynamics in order to link the respective one-period spreads. This will be done in the following section.

## 2.2 The multi-period forward Fair Tranche Spread

The aim of this section consists in deriving an extension to the one-period model in order to generalize the forward spread dynamics to a multi-period tenor. This allows for the pricing of "real world" deals:

**Lemma 2.4.** *Again consider a deal with tenor $[T_a, T_b]$ and tranche $[A, B]$. The forward multi-period spread dynamics with the same tenor, note $S^{A,B}_{a,b}$, can be written as*

$$\frac{dS^{A,B}_{a,b}(t)}{S^{A,B}_{a,b}(t)} = (\Lambda(t) + \varsigma(t))\,\rho\,(\Lambda(t))'\,dt - (\Lambda(t) + \varsigma(t))\,dZ_t \tag{74}$$

*with*

$$\Lambda(t) = \sum_{i=a+1}^{b} \frac{\delta_i A(t,T_i) Y_i(t)}{\hat{C}^{A,B}_{a,b}(t)} \gamma_i(t) \tag{75}$$

$$\varsigma(t) = \frac{A(t,T_b)Y_b(t)}{A(t,T_a)Y_a(t) - A(t,T_b)Y_b(t)} \gamma_b(t) \tag{76}$$

*Proof.* Recall that the associated numeraire is given by:

$$\hat{C}^{A,B}_{a,b}(t) = \sum_{i=a+1}^{b} \delta_i A(t,T_i) E^t_{Q^{T_i}}\left[X(T_i)\right] \tag{77}$$

$$= \sum_{i=a+1}^{b} \delta_i A(t,T_i) Y_i(t) \tag{78}$$

$A$ as well as $Y_i$ follow the respective dynamics:

$$\frac{dA(t,T_i)}{A(t,T_i)} = r_t dt \tag{79}$$

and

$$\frac{dY_i(t)}{Y_i(t)} = \gamma_i(t) dZ_t \tag{80}$$

Hence the dynamics of the numeraire itself can be written as :

$$d\hat{C}^{A,B}_{a,b}(t) = \sum_{i=a+1}^{b} \delta_i d\left(A(t,T_i) Y_i(t)\right) \tag{81}$$

$$= \sum_{i=a+1}^{b} \delta_i A(t,T_i) Y_i(t) \left(r_t dt + \gamma_i(t) dZ_t\right) \tag{82}$$

$$= \hat{C}^{A,B}_{a,b}(t) r_t dt + \left( \sum_{i=a+1}^{b} \delta_i A(t,T_i) Y_i(t) \gamma_i(t) \right) dZ_t \tag{83}$$

This gives

$$\frac{d\hat{C}^{A,B}_{a,b}(t)}{\hat{C}^{A,B}_{a,b}(t)} = r_t dt + \left( \sum_{i=a+1}^{b} \frac{\delta_i A(t,T_i) Y_i(t)}{\hat{C}^{A,B}_{a,b}(t)} \gamma_i(t) \right) dZ_t \tag{84}$$

$$= r_t dt + \left( \sum_{i=a+1}^{b} w_i(t) \gamma_i(t) \right) dZ_t \tag{85}$$

$$= r_t dt + \Lambda(t) dZ_t \tag{86}$$

Hence one can write:

$$\frac{\hat{C}_{a,b}^{A,B}(T_a)}{\hat{C}_{a,b}^{A,B}(t)} = \exp\left[\int_t^{T_a}\left(r_s - \frac{1}{2}\Lambda(s)\right)ds + \int_t^{T_a}\Lambda(s)dZ_s\right] \tag{87}$$

With $Q_{a,b}^{A,B}$ representing the probability measure associated to the numeraire $\hat{C}_{a,b}^{A,B}(t)$ we write:

$$\left.\frac{d\hat{Q}_{a,b}^{A,B}}{dQ}\right|_{\mathcal{F}_t} = \frac{\hat{C}_{a,b}^{A,B}(T_a)N^{Q^t,t}(t)}{\hat{C}_{a,b}^{A,B}(t)N^{Q^{T_a},t}(T_a)} \tag{88}$$

$$= \frac{\hat{C}_{a,b}^{A,B}(T_a)A(t,T_a)}{\hat{C}_{a,b}^{A,B}(t)} \tag{89}$$

$$\tag{90}$$

With Girsanov this gives:

$$d\hat{Z}_t^{a,b} = dZ_t - \rho\left(\Lambda(t)\right)' dt \tag{91}$$

As in the one-period case we have to derive the dynamics of the forward spread $S_{a,b}^{A,B}$. With Itô's formula this amounts to:

$$\frac{d\left(\frac{1}{\hat{C}_{a,b}^{A,B}(t)}\right)}{\frac{1}{\hat{C}_{a,b}^{A,B}(t)}} = \left(\Lambda(t)\rho\left(\Lambda(t)\right)' - r_t\right)dt - \Lambda(t)dZ_t \tag{92}$$

Further more,

$$d\left(A(t,T_k)Y_k(t)\right) = A(t,T_k)dY_k(t) + Y_k(t)dA(t,T_k) \tag{93}$$

$$= A(t,T_k)Y_k(t)\left(r_t dt + \gamma_k(t)dZ_t\right) \tag{94}$$

Considering the Leibniz rule, one can derive the following dynamics:

$$d\left(\frac{A(t,T_k)Y_k(t)}{\hat{C}_{a,b}^{A,B}(t)}\right) = A(t,T_k)Y_k(t)d\left(\frac{1}{\hat{C}_{a,b}^{A,B}(t)}\right) + \frac{1}{\hat{C}_{a,b}^{A,B}(t)}d\left(A(t,T_k)Y_k(t)\right) \tag{95}$$

$$+ d\left(A(t,T_k)Y_k(t)\right)d\left(\frac{1}{\hat{C}_{a,b}^{A,B}(t)}\right) \tag{96}$$

$$= \frac{A(t,T_k)Y_k(t)}{\hat{C}_{a,b}^{A,B}(t)}\left[\left(\Lambda(t)\rho\left(\Lambda(t)\right)' - r_t\right)dt - \Lambda(t)dZ_t + r_t dt + \gamma_k(t)dZ_t\right. \tag{97}$$

$$\left.-\gamma_k(t)\rho\left(\Lambda(t)\right)' dt\right] \tag{98}$$

$$= \frac{A(t,T_k)Y_k(t)}{\hat{C}_{a,b}^{A,B}(t)}\left[\left(\Lambda(t) - \gamma_k(t)\right)\rho\left(\Lambda(t)\right)' dt + \left(\gamma_k(t) - \Lambda(t)\right)dZ_t\right] \tag{99}$$

$$= \frac{A(t,T_k)Y_k(t)}{\hat{C}_{a,b}^{A,B}(t)}\left(\gamma_k(t) - \Lambda(t)\right)\left(dZ_t - \rho\left(\Lambda(t)\right)' dt\right) \tag{100}$$

$$= \frac{A(t,T_k)Y_k(t)}{\hat{C}_{a,b}^{A,B}(t)}\left(\gamma_k(t) - \Lambda(t)\right)d\hat{Z}_t^{a,b} \tag{101}$$

The dynamics presented above hold $\forall k \in [a;b]$.

However for $k = a$ things get simpler as $\gamma_a(t) = 0$. Thus we obtain :

$$d\left(A(t,T_a)Y_a(t)\right) = A(t,T_a)Y_a(t)r_t dt \tag{102}$$

$$d\left(\frac{A(t,T_a)Y_a(t)}{\hat{C}^{A,B}_{a,b}(t)}\right) = \frac{A(t,T_a)Y_a(t)}{\hat{C}^{A,B}_{a,b}(t)}\left(-\Lambda(t)\right)d\hat{Z}^{a,b}_t \tag{103}$$

Recall that the forward spread is given by:

$$S^{A,B}_{a,b}(t) = \frac{\sum_{i=a+1}^{b} A(t,T_{i-1})Y_{i-1}(t) - A(t,T_i)Y_i(t)}{\hat{C}^{A,B}_{a,b}(t)} \tag{104}$$

This reduces to

$$S^{A,B}_{a,b}(t) = \frac{A(t,T_a)Y_a(t) - A(t,T_b)Y_b(t)}{\hat{C}^{A,B}_{a,b}(t)} \tag{105}$$

The dynamics follow

$$dS^{A,B}_{a,b}(t) = \left[\frac{A(t,T_a)Y_a(t)}{\hat{C}^{A,B}_{a,b}(t)}\left(-\Lambda(t)\right) - \frac{A(t,T_b)Y_b(t)}{\hat{C}^{A,B}_{a,b}(t)}\left(\gamma_b(t) - \Lambda(t)\right)\right]d\hat{Z}^{a,b}_t \tag{106}$$

$$= \left[-\Lambda(t)S^{A,B}_{a,b}(t) - \frac{A(t,T_b)Y_b(t)}{\hat{C}^{A,B}_{a,b}(t)}\left(\gamma_b(t) - \Lambda(t)\right)\right]d\hat{Z}^{a,b}_t \tag{107}$$

Finally we obtain

$$\frac{dS^{A,B}_{a,b}(t)}{S^{A,B}_{a,b}(t)} = \left[-\Lambda(t) - \frac{A(t,T_b)Y_b(t)}{A(t,T_a)Y_a(t) - A(t,T_b)Y_b(t)}\gamma_b(t)\right]d\hat{Z}^{a,b}_t \tag{108}$$

$$= \left(-\Lambda(t) - \varsigma(t)\right)d\hat{Z}^{a,b}_t \tag{109}$$

$$= -\left(\Lambda(t) + \varsigma(t)\right)\left(dZ_t - \rho\left(\Lambda(t)\right)' dt\right) \tag{110}$$

$$= \left(\Lambda(t) + \varsigma(t)\right)\rho\left(\Lambda(t)\right)' dt - \left(\Lambda(t) + \varsigma(t)\right)dZ_t \tag{111}$$

□

Finally one can express the multi-period spread dynamics in function of one-period spreads:

$$S^{A,B}_{a,b}(t) = \frac{\sum_{i=a+1}^{b} \delta_i A(t,T_i)Y_i(t) \times S^{A,B}_{i-1,i}}{\hat{C}^{A,B}_{a,b}(t)} \tag{112}$$

$$= \sum_{i=a+1}^{b} w_i(t) \times S^{A,B}_{i-1,i} \tag{113}$$

with

$$w_i(t) = \frac{\delta_i A(t,T_i)Y_i(t)}{\hat{C}^{A,B}_{a,b}(t)}$$

The formula above formalizes the central idea of this article and reveals the relationship between one-period forward spreads. This allows for pricing of deals with any maturity.

# 3 Conclusion

In this paper, we have suggested a multi-period extension of the Market Model for Options on synthetic CDS Index Tranches. CDS Index Tranches are synthetic CDO tranches with standardized characteristics which encourages liquidity. This is the most important issue when dealing with Market Models in general. Thus a CDO option Market Model only makes sense on this specific credit derivatives class.
But in what consist the advantages of a market model for forward start options on CDS Index tranches? Why not just use the popular Black & Scholes formula for pricing?

***First***, there are liquidity related advantages. Suppose that an investor had a position on an option with exercise period (tenor) $[T_c; T_d]$. However the most liquid option type is the one with exercise period $[T_a; T_b]$. In order to calibrate his option value on the market, the investor can reconstruct his proper exercise period with multi-period forward spread rates which themselves are calibrated via one-period spread rates to the most liquid tenor $[T_a; T_b]$ (when we suppose all other option characteristics being equal). This results in more realistic market price. Further when calibrating to several tenors one takes into account different market views on the evolution of credit risk, i.e. the view is not static anymore. This finally allows for considering the vega when it comes to pricing of credit derivatives, i.e; the investors pays for the volatility.

***Second***, a Market Model sometimes is the only way for pricing some options: Suppose a forward start CDO option with tenor $[T_a; T_b]$ on a synthetic CDO tranche with standard tranche size based on a CDS Index series. However the deal will be set up just at $t = T_a$ (Issue Date $= T_a$). This means that the forward Fair Tranche Spread will be determined in $t = T_a$. Hence the forward Fair Tranche Spread is no longer a martingale and we have to price this option by calculating expectations of the forward dynamics.

***Finally***, dynamic modeling is the most suited approach when it comes to evaluating more complex deals. More and more OTC combo deals have been coming up recently. In order to limit ramp up costs, the issuer (structurer) is interested in choosing a liquid and hence standardized basis for his deal. One can imagine options on CDS index tranches structured by combo or kicker components which complicate the pay-off. In these cases the modeling of the spread dynamics themselves is a real advantage.

# 4 Implementation

## 4.1 General Behavior

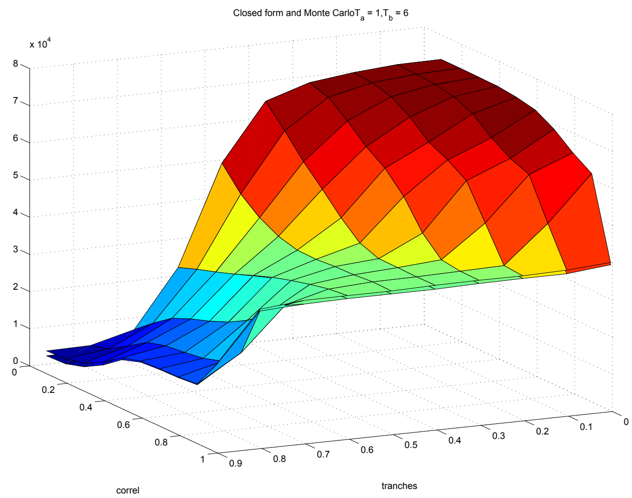


Figure 1: Call Value Surface.

In figure 1 we assume that a CDO deal is set up in the future (at $T_a$). By definition, the risky equity tranches pay the highest spread. Thus the call value is at its maximum. Note that with growing time distance until the deal set-up, the risk associated to junior tranches is growing. This is due to growing uncertainty when it comes to forecasting idiosyncratic risk. This leads to the pronounced "plateau" of the surface for junior positions. The plateau has a triangular form as with increasing correlations the equity tranches become less risky (they are short correl).

For mezzanine tranches we have a 3-dimensional settle point which is the consequence of the fact that these tranches are long/short correl at the same time.
As expected senior tranches pay the lowest spread and thus generate the lowest call value. However note that their surface is upward sloping due to their long correl behavior: when correlation increases, systemic risk gets more and lore pronounced which makes senior tranches riskier. As a consequence their Fair Spread increases.

## 4.2 Comment

As expected one can generally state that the market model furnishes lower spreads as its closed-form counterpart. This is due to more inputs when calibrating and different probability filtration. However this general tendency amplifies the more we move towards seniority. Senior tranches are more exposed to systemic risk which is less volatile. As the market model takes into account the Vega, the tranche is less risky. The most important difference to the closed form solution is obtained with very senior tranches and low correlation: here again senior tranche are less sensitive to idiosyncratic risk variations which is taken into account by the model presented in this article.

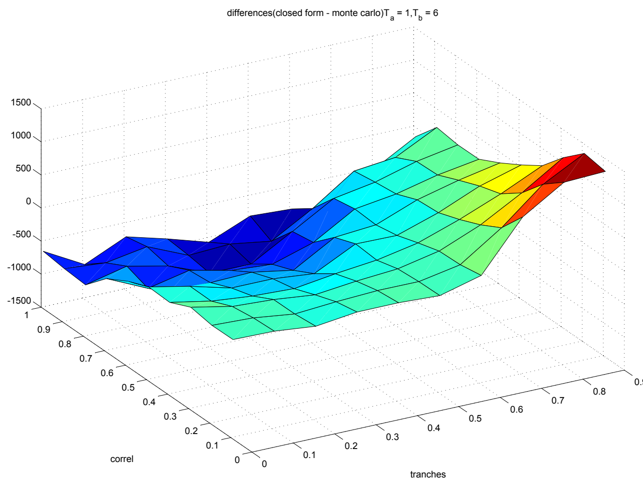


Figure 2: Differential Closed-form formula vs. Dynamics.

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