

# Hedging Interest Rate Margins on Demand Deposits

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This paper deals with risk mitigation of interest rate margins related to a bank's demand deposits. We assume the demand deposit evolution to be related to both interest rates and some exogenous factor which can be interpreted as business risk or model risk. We subsequently discuss the trade-off between the alleviation of interest rate risk and the excess return due to investing in longer term assets. We take the viewpoint of an asset and liability manager, dealing with interest rate derivatives and focusing on the bank's net operating income. We firstly derive static hedging strategies in a mean-variance framework. Then we compare them with dynamic investment strategies. Applying Duffie and Richardson's (1991) approach, we firstly study the case of demand deposits bearing a rate which is an affine function of the market rate. We prove that the optimal hedging risk profile involves a semi-static replication of Libor exotic options, thus revealing the hidden optionality in demand deposits and paving the way for low transaction cost hedging strategies. Finally, we derive dynamic optimal strategies when the deposit rate follows some more complex function of the market rate and we show their robustness with respect to other risk measures.

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## 1. Introduction

Under IFRS – the new 2005 international accounting standards – banks account demand deposits at amortized cost<sup>1</sup>. Moreover, the US Securities and Exchange Commission asks American banks to report in annual (10-K) and quarterly (10-Q) documents, indicators concerning interest rate margins and their sensibilities to interest rate shocks. However, in internal processes, many banking establishments perform the computation of full fair-value indicators for the market value of equity. As for demand deposits, the well-known fair-value approach developed for example in Hutchison and Pennacchi (1996) and Jarrow and van Deventer (1998) has been the main way to assess demand deposits so far. However, from the

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<sup>1</sup> The IFRS actually ask banking establishments to account demand deposits at a Fair Value equal to their nominal value, which is equivalent. See e.g. IAS 39 – Measurement – Subsequent Measurement of Financial Liabilities – Official IASB Website <http://www.iasb.org/>

viewpoint of Asset and Liability Management, banking establishments experience great difficulty with it, due to significant sensitivity towards the choice of the model. Consequently banks, as well as financial analysts and banking managers, pay more and more attention to the demand deposit income at amortized cost.

Indeed, a worldwide study of the Bank for International Settlements (see English (2002)) shows that risk mitigation in interest rate margins has been a significant concern for banks during these last twenty years, before the subprime crisis. Since then, as a first step, the European Commission endorsed in November 2007 the IAS 39 Fair Value Option Amendment and two carve outs, allowing hedging strategies that provide a smoothed income on demand deposits (Carved-Out Fair Value Hedge)<sup>2</sup>. The IASB and the FASB are now leading joint reflections in order to replace the latter carve outs by a new kind of hedging strategy, the Interest Margin Hedge (IMH) (see e.g. Adam (2007)), which aims at assessing the volatility of demand deposits' interest rate margins rather than the volatility of their fair value. In this paper, we propose an approach based upon the interest rate margin of demand deposits for a given time period. For simplicity, we essentially put the stress on demand deposits rather than on the balance sheet as a whole.

Jarrow and van Deventer (1998) assume the demand deposit amount to be contingent to interest rates only, considering interest rate margins and the related full fair-value as pure exotic interest rate derivatives. In such a framework, the market is complete, thus the risk neutral measure used for valuation is unique.

However, as stated in Kalkbrener and Willing (2004), the demand deposit amount is not only contingent to interest rates: it carries some business risk orthogonal to market risk. Whenever this business risk is related with various macroeconomic risks, one may think of some risk premium being involved in the discount rate while in the complete market case, one can compute expected discounted payoff under the risk neutral measure as it is usually the case for interest rate derivatives. However, Ho and Saunders (1981) and later Wong (1997) and Saunders and Schumacher (2000) show that not only market rates but also the regulatory framework, the bank's market structure and its credit risk exposure for example may influence net interest margins and interest rate margins as a consequence.

As a first approach, we propose to derive interest rate derivative-based optimal static strategies to mitigate the risk carried by the interest rate margin. Then, our concern is to improve the hedging quality thanks to dynamic strategies. Due to the incomplete market framework in which we assess interest rate margins, the risk-neutral viewpoint is not unique and, similarly to Kalkbrener and Willing (2004) we propose to deal with this issue thanks to variance-minimal hedging techniques. The choice of a mean-variance framework is mainly due to this analytical tractability. Moreover, it leads to additive strategies with respect to the choice of the balance sheet item, paving the way for optimal hedging strategies for the whole balance sheet. Duffie and Richardson (1991) derive explicit dynamic hedging strategies in a framework where both the underlying asset and the asset to hedge follow a geometric Brownian diffusion process. Indeed, such techniques can be used when deposit rates are affine functions of market rates, as in Hutchison and Pennacchi (1996) and Jarrow and van Deventer (1998).

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<sup>2</sup> See European Commission's (EC) Reference Document IP/04/1385 – Official EC Website <http://ec.europa.eu>

However when dealing with more general modeling of the net interest margin, we propose to use the *hedging numéraire* theory developed in Gouriéroux, Laurent and Pham (1998) or Pham, Rheinländer and Schweizer (1998). Indeed, this theory extends Duffie and Richardson's (1991) approach, since it allows us to derive explicit dynamic strategies to hedge the interest rate margin for a given time period. Then we check the reliability of the corresponding optimal strategies and their robustness, by considering the risk mitigation associated with asymmetric risk measures like the Expected Shortfall.

This paper is organized as follows. In Section 2 we show how interest rate margins have become a major point of concern for banking establishments today, with a focus on the US case. In Section 3 we propose a modeling framework for demand deposits, interest rates and the interest rate margin. In Section 4 we set the optimization problem and derive static and dynamic strategies to hedge the interest rate margin for a given time period. In Section 5 we give empirical facts about these strategies and propose to compare their performances. We show that comparing with the static ones, dynamic hedging strategies better account for effects due to demand deposits' specific risk. We also show some robustness of optimal dynamic strategies when switching to other risk measures like the Value-at-Risk and the Expected Shortfall.

## 2. Motivating Interest Rate Margins

In the 1990's, many studies focused on the fair value of demand deposits within a bank. For example, this approach is officially recommended and developed in the Office of Thrift Supervision's official publication about the *Net Portfolio Value Model* (1994)<sup>3</sup>. Since the demand deposits' fair value is set as the discounted sum of future cash flows on demand deposits, its computation requires an assessment of future interest rate margins. This is what we especially observe in Selvaggio (1996) and Hutchison and Pennacchi (1996). Later, Jarrow and van Deventer (1998) and O'Brien (2000) come to the same result under some arbitrage-free framework for the valuation of demand deposits.

With the adoption of the IFRS in 2005, regulators pay increasing attention to interest rate margins. Banks' quarterly (10-Q) and annual (10-K) reports to the SEC<sup>4</sup> contain specific sections about their net interest incomes and the related sensitivities within 1 year horizon, towards standardized interest rate shocks. These shocks are usually +/- 200bp interest rate gradual shocks during the upcoming year<sup>5</sup>.

We have been collecting data concerning the net interest income and its sensitivity for 20 US banking establishments<sup>6</sup> of almost the same asset size and featuring a similar 'involvement in retail banking'. As for the latter point, we took the number of branches within the United States as an indicator of the involvement in retail banking activities: thus, each of these establishments feature between 179 and 1711 branches within the United States. They also seem close as for the ratios net interest income / asset size and number of agencies / asset size. This latter point shows a quite similar involvement in retail banking activities (cf. Appendix A).

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<sup>3</sup> See OTS Official Website – <http://www.ots.treas.gov/>

<sup>4</sup> See e.g. Item 7 'Management's Discussion and Analysis of Financial Condition and Results of Operations' in 10-K reports, and the corresponding Item 2 in 10-Q reports.

<sup>5</sup> See Item 7A 'Quantitative and Qualitative Disclosures About Market Risk' in 10-K reports and the corresponding Item 3 in 10-Q reports.

<sup>6</sup> See Appendix A: List of US Banks used in interest rate margins analysis.

During year 2005 Libor rates have been – almost gradually – increasing by nearly 200 basis points, closely reproducing interest rate scenarios recommended by the SEC, as stated above. Thus we have been measuring the predicting power of the computed sensitivities for the upcoming year<sup>7</sup>. To achieve that, we examine how the *ex-post* variations of the net interest incomes with respect to their previsions in some central interest rate scenario differ from the sensitivities displayed *ex-ante* in SEC reports. The coefficient of 1.37 in Table 2.1 below shows that the sensitivity computed *ex-ante* follows but slightly underestimates reality. Moreover, because of the weakness of the R-square and the F-statistic being far beyond its critic value, the explanative power of the *ex-ante* sensitivity seems pretty limited.

	Intercept (Standard Deviation)	Ex-ante Sensitivity (StDev)	R <sup>2</sup>	F-statistic (Critic Value)
Net Interest Income Variation with respect to central IR scenario	8.32% (2.43%)	1.37 (1.05)	29%	1.72 (0.21)

**Table 2.1. Net Interest Income Variation: Ex-Ante vs. Ex-Post.**

The Net Interest Income Variation with respect to central IR scenario stands for the relative difference between the net interest income observed during year 2005 (*ex-post*) and the income that could be expected, according to some central interest rate scenario. It is regressed upon the Ex-ante Sensitivity which is the same variation, computed *ex-ante* by banking establishments at the beginning of year 2005, using complex simulation methods. This is the variation in the net interest income subsequent to a gradual linear rise of 200 basis points in interest rates during the upcoming year, with respect to what it would be in some central interest rate scenario.

Actually, this fact is not surprising, since banks' disclaimers in SEC reports already warn us about other factors that could damage the explanatory power of the computed sensitivities. Indeed we can easily imagine that for example amounts in assets and liabilities are not meant to evolve only in relation with interest rates and that the net interest income may carry some convexity towards interest rates, due to embedded options. Moreover, the computation of sensitivities may suffer from some heterogeneity among banking establishments.

Moreover, in a study for the BIS, English (2002) shows that banks have been avoiding significant exposures of the interest rate margin to market interest rates (short and long term) although we still notice slight sensitivities towards the yield curve slope in some European countries (Germany, Norway, Switzerland and Sweden). This shows that banks pay significant interest to the risk carried by interest rate margins.

### 3. Modeling Framework

#### 3.1. Market Rates

We consider some time horizon  $T$  such that we deal with the corresponding quarterly interest rate margin. Besides, we assume that the forward Libor rate at horizon date  $T$  for the time period  $\delta T > 0$  of the interest rate margin follows a Libor Market Model, as defined in Brace, Gatarek and Musiela (1997) and Miltersen, Sandmann and Sondermann (1997):

$$dL_t = L_t(\mu_L dt + \sigma_L dW_L(t)), \quad (1)$$

where we denote  $L_t = L(t, T, T + \delta T)$ .

<sup>7</sup> Let us remark that the related time period (2005-2006) is located way before the 2007 subprime crisis.

For model simplicity,  $\mu_L$  and  $\sigma_L$  are assumed to be constant and deterministic and we denote the related interest rate risk premium by  $\lambda = \frac{\mu_L}{\sigma_L}$ . We will thereafter be able to account for greater average returns when investing in long term bonds than in short-term assets (see e.g. Chapter 11 in Campbell, Lo and MacKinlay (1997)).

### 3.2 Demand Deposit Amount

We assume that the demand deposit amount follows:

$$dK_t = K_t (\mu_K dt + \sigma_K d\bar{W}_K(t)) \quad (2)$$

where  $\bar{W}_K$  is a standard Brownian motion. For simplicity, the trend  $\mu_K$  and the volatility  $\sigma_K$  are assumed to be constant and deterministic, though the results readily extend to the time-dependant and deterministic case, for instance to deal with seasonal effects. Clearly, the trend and volatility terms depend upon the liabilities being considered. We use a Brownian motion to cope with the evolution of the amount of demand deposits, for a matter of tractability.

At a given time, the outstanding nominal amount results from cash inflows and withdrawals from existing clients, including account cancellations. This point of view is rather related to the fair value of non-maturing deposits. On the other hand, one could include the net cash-flows resulting from the opening of new accounts, which could come either from endogenous growth or external development. This can be associated with the “embedded value” in the insurance terminology and should rather be the point of view of stockholders. The IAS 39 facilitates the fair value hedge of deposits, while taking in account the embedded value of deposits would rather lead to the hedge of the interest rate margins (cash-flow hedge).

The following tables provide maximum likelihood estimations of  $\mu_K$  and  $\sigma_K$  in a number of cases. We considered monthly amounts issued from the American, European (Euro Zone) and Japanese markets between January 1999 and September 2007. We collected:

- amounts of each market’s M2 monetary aggregates – excluding currency in circulation (M0) – endowing overnight deposits, check accounts, savings and certificates of deposit of agreed maturity up to 2 years, as defined by central banks;
- amounts of each market’s M1 monetary aggregate, excluding currency in circulation ; this aggregate endows only overnight deposits and check accounts.

Table 3.2 contains estimations for two submarkets in the Euro Zone – France and Germany – showing very little transfer effects from a submarket to another (overall and submarket’s volatilities being close) but a more significant growth (9.24%) in the overall market due to the inclusion of new countries in the Euro Zone during the estimation period. This phenomenon can be compared to a bank’s establishment external growth policy. Finally, Table 3.3 contains parameter estimations for two examples of emerging markets – Turkey and Ukraine – showing the tremendous growth of such markets during the last decade.

We also notice that aggregates containing both savings and sight deposits (M2 and assimilated) feature greater stability than those containing only demand deposits (M1). Indeed, there exist money transfers between the different types of accounts, generating volatility on the M1 aggregate while the aggregate including saving accounts remains stable. Of course, this observation strongly depends on the various types of deposits that banks

propose to their customers, on each marketplace. For example, in the US, clients often own several types of accounts (MMDA, NOW, checkable accounts, etc.) in addition to asset management services, which feature significant transaction costs or heavy tax conditions, thus not as convenient as usual deposits.

Market	Monetary aggregate	$\mu_K$	$\sigma_K$
US	Demand Deposits	-2.29%	8.24%
US	Demand and Checkable Deposits	-0.31%	5.16%
US	M2 - M0	5.99%	1.30%
Euro Zone	Demand Deposits	9.24%	6.08%
Euro Zone	M2-M0	6.27%	2.33%
Japan	M2-M0	2.83%	2.26%

**Table 3.1. Estimation of Demand Deposit Parameters for US, Euro Zone and Japan's Monetary Aggregates.**

Estimation Period: January 1999 to September 2007. Sources: Federal Reserve Bank of St. Louis (<http://www.stlouisfed.org>), European Central Bank (<http://www.ecb.int/>) and Bank of Japan (<http://www.boj.or.jp/en/>). The estimations are all given on a yearly basis.

Market	Monetary aggregate	$\mu_K$	$\sigma_K$
Euro Zone	Demand Deposits	9.24%	6.08%
France	Demand Deposits	5.93%	5.77%
Germany	Demand Deposits	8.47%	6.19%
Euro Zone	M2-M0	6.27%	2.33%
Germany	M2-M0	3.21%	1.63%

**Table 3.2. Estimation of Demand Deposit Parameters for Euro Zone and Submarkets (France, Germany).**

Source: European Central Bank (<http://www.ecb.int/>), Banque de France (<http://www.banque-france.fr/>) and Deutsche Bundesbank (<http://www.bundesbank.de/>). The estimations are all given on a yearly basis.

Market	Monetary aggregate	$\mu_K$	$\sigma_K$
Turkey	M1 - M0	37.93%	35.97%
Turkey	M2 - M0	33.63%	11.00%
Ukraine	M1 - M0	33.41%	13.45%
Ukraine	M2 - M0	36.68%	9.12%

**Table 3.3. Estimation of Demand Deposit Parameters for Some Emerging Markets (Turkey, Ukraine).**

Sources: Central Bank of Republic of Turkey<sup>8</sup> (<http://www.tcmb.gov.tr/yeni/eng/>) and National Bank of Ukraine (<http://www.bank.gov.ua/ENGL/>). The estimations are all given on a yearly basis.

### 3.3 Linking Deposit Amount and Interest Rates

As in Kalkbrener and Willing (2004) we assume the dynamics of the demand deposit amount to be correlated with interest rates:

<sup>8</sup> Data are available on the Internet at <http://www.tcmb.gov.tr/yeni/eng/>

$$d\bar{W}_K(t) = \rho dW_L(t) + \sqrt{1 - \rho^2} dW_K(t) \quad (3)$$

where  $W_K$  is a Brownian motion orthogonal to  $W_L$ , and  $\rho$  some constant and deterministic correlation factor. Let us emphasize that, like in Fraundorfer and Schurle (2003), the demand deposits may feature other sources of risk than the one related to interest rates. Then  $W_K$  can be considered as some component independent from interest rates movements. The latter approach enlightens our previous study on US banks' net interest incomes. Let us remark that, when setting  $\rho = 1$ , we fall into the complete markets framework of Jarrow and van Deventer (1998). More precisely, we might get close to their framework when setting directly

$$\frac{dK_t}{K_t} = k_d dt + k_v \frac{dL_t}{L_t} \quad (4)$$

with  $k_d = \mu_K$  and  $k_v = \frac{\rho\sigma_K}{\sigma_L}$ , thus neglecting the deposits' specific risk. We can eventually

notice that Jarrow and van Deventer (1998) or Hutchison and Pennacchi (1996) propose different modelings as for the deposit amount process, under which the derivation of optimal hedging strategies is usually not feasible unless we neglect the specific effects of business risk.

The correlation between the variations of demand deposit amount and that of interest rates can be related to money transfers between deposit accounts and other types of deposits. Janosi, Jarrow and Zullo (1999) refer to this phenomenon as *disintermediation*. They estimate the correlation parameter using bank data coming from the Federal Reserve Bulletin for various types of accounts – namely Negotiable Orders of Withdrawal (NOW), passbooks, statement and demand deposit accounts. Their study exhibits negative values for all account types, causing demand deposit amounts fall when short rates rise.

We refer to the Engle and Granger method detailed in Ericsson and MacKinnon (1999) to estimate the correlation parameter between the deposit amount and the market rate. Janosi et al. (1999) use a very similar method, although they also pay attention to autocorrelation and short term effects. We show our results in Tables 3.4 and 3.5.

Market	Monetary Aggregate	Related Market Rate	$\sigma_L$	$\sigma_K$	$\rho$
US	Demand Deposits	USD 3M Libor	21.80%	8.24%	0%
US	Demand and Checkable Deposits	USD 3M Libor	21.80%	5.16%	-11.28%
Euro	Demand Deposits	3M Euribor	15.42%	6.08%	-70.85%

**Table 3.4. Estimation of Correlation Parameter for US and Euro Zone's Demand Deposits.**

The estimations of the volatility parameters are given on a yearly basis.

Market	Monetary Aggregate	Related Market Rate	$\sigma_L$	$\sigma_K$	$\rho$
Euro	Demand Deposits	3M Euribor	15.42%	6.08%	-70.85%
France	Demand Deposits	3M Euribor	15.42%	5.77%	-46.70%
Germany	Demand Deposits	3M Euribor	15.42%	6.19%	-84.65%

**Table 3.5. Estimation of Correlation Parameter for Euro Zone's and Some Submarkets' Demand Deposits.**

The estimations of the volatility parameters are given on a yearly basis.

We did not show any results for M2-type monetary aggregates, since they do not exhibit significant correlation with interest rates. Indeed the transfers between saving accounts and

more elaborate investment schemes may not be driven by interest rates only. There are probably additional factors like transaction costs and tax charges, that drive customers' arbitrages between M2-type deposits and time deposits or asset management investment opportunities.

The preceding estimators of the correlation parameter do not integrate lag and short term effects either but may be analysed though. We notice that they vary significantly from one aggregate to another and when switching from a marketplace to its submarkets. This may also occur among individual banking establishments and their subsidiaries, and when modifying the perimeter among demand deposits. In the US, demand deposit volatility is higher (8.24% vs. 5.16%) but almost not correlated with interest rate variations, comparing to demand and checkable deposits. Conversely, in the Euro Zone, most of the demand deposit amount volatility (-70.85%) seems to be due to interest rate variations.

### 3.4. Deposit Rate Modeling and Interest Rate Margin

Depending on the local business model, deposit accounts may bear interests for clients. As suggested by Hutchison (1995), Hutchison and Pennacchi (1996) or Jarrow and van Deventer (1998), the deposit rate may exhibit some dependence with respect to market rates.

Since we assess the interest rate margin at some fixed horizon  $T$ , we only deal with the deposit rate at this date. Hutchison and Pennacchi (1996) assume the deposit rate to fulfil some affine relation with the market rate and the residuals to be linked with the deposit amount's elasticity, thanks to some equilibrium model developed in Hutchison (1995). Indeed, for example, when we perform a linear regression of the US M2 own rate upon the 3-month Libor rate, the residuals feature a correlation of  $-10\%$  with M2's growth<sup>9</sup>. This is possible to derive optimal hedging strategies in the case where the deposit rate is an affine function of the Libor rate which features a residual term correlated with the deposit amount's growth  $\overline{W}_K$ . Indeed, the optimal hedging strategies we derive in section 4 are linear with respect to the interest rate margin. However, from now on, we assume the deposit rate to be a deterministic function  $g$  of the market rate  $L_T = L(T, T, T + \delta T)$ .

The graphs below (see Figure 3.6) confirm the intuition of some affine dependence between the deposit rate and the market rate.

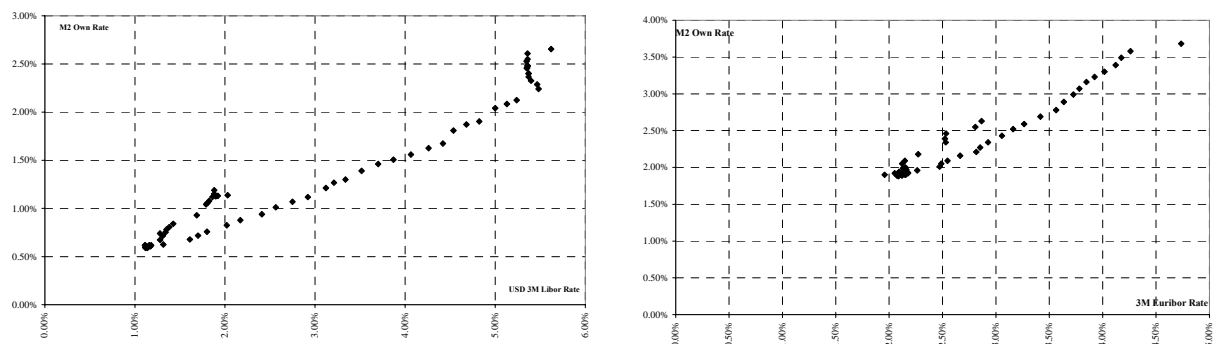


Figure 3.6. Deposit rate and market rate in US (left) and Euro Zone (right).

<sup>9</sup> This estimation is significative at 1% confidence level, according to the Fisher's zero correlation test (see e.g. Campbell et al. (1997)).

The scatters represent the deposit rate on the Y axis and the market rate on the X axis. Data sample period ranges from January 2002 to September 2007.

Table 3.7 contains the estimations of  $\alpha$  and  $\beta$  corresponding to the linear regression  $g(L_T) = \alpha + \beta L_T + \varepsilon_T$ . We focus on long term effects only thanks to Engle and Granger's method (see e.g. Ericsson and MacKinnon (1999)), though Jarrow et al. (1999) and Hutchison and Pennacchi (1996) deal more thoroughly with lag and short term effects. Using data on individual bank retail deposit interest rates, Hutchison and Pennacchi (1996) find a  $\beta$  coefficient equal to 0.40 for NOW and 0.83 for MMDA. As Table 3.7 shows, our estimations are located in this range.

Market	Monetary Aggregate	Market Rate	$\alpha$ (Intercept)	$\beta$ (Market Rate)	R <sup>2</sup>
US	M2 - M0	USD 3M Libor	-0.41%	0.66	86%
Euro Zone	M2 - Enterprises and Households	3M Euribor	0.46%	0.69	96%

**Table 3.7. Estimating relationship between deposit rate and market rate in the US and in the Euro Zone.** The estimation period for the US is Nov. 1986 – Sept. 2007 and for the Euro Zone, Jan. 2003 – Sept. 2007. The alpha parameter is given on a yearly basis.

The case of the Japanese market differs from the US and the Euro Zone. Indeed, between June 2001 and February 2006, market rates were very low (below 0.08% on a yearly basis), compelling banks with shrinking dramatically deposit rates in order to keep positive margins. Indeed deposit rates were very close to zero during this period. We propose a focus on these facts in the following table.

Explanatory Variables		Coefficient	Standard Error	T-statistic	P-value
When Libor	Intercept	0.004%	0.027%	0.17	87%
Rate < 0.08%	3M Libor Rate	0.546	0.444	1.23	22%
When Libor	Intercept	0.045%	0.006%	7.89	<<1%
Rate > 0.08%	3M Libor Rate	0.387	0.015	25.24	<<1%

**Table 3.8. Japanese Deposit Rate Modeling.**

We gathered deposit and market rates from April 1999 to August 2007 at a monthly frequency. We estimate the following model:  $CR_t = \mathbf{1}_{L_t < 0.08\%} (\alpha_1 + \beta_1 L_t) + \mathbf{1}_{L_t > 0.08\%} (\alpha_2 + \beta_2 L_t) + \varepsilon_t$  where  $CR_t$  (resp.  $L_t$ ) is the deposit (resp. market) rate at date  $t$ . Italic values (coefficients when Libor Rate < 0.08%) are non significant at 5% probability level. (Source: Bank of Japan). Estimations are given on a yearly basis.

Hence, we will further focus on the two following sub-cases as for the modeling of deposit rates:

$$g(L_T) = \alpha + \beta L_T \text{ (US and Euro Zone case)} \quad (5a)$$

and

$$g(L_T) = \alpha + \beta L_T \text{ (Japanese case).} \quad (5b)$$

Finally, the interest margin measures the income related to the shift between interest rates concerning demand deposits.

*Definition 3.1. (Interest Rate Margin) The Interest Rate Margin at date T stands for the cash-flow generated upon the time period  $[T, T + \delta T]$  by the investment of the amount of demand deposits on the short term rate  $L_T$  minus the interests  $g(L_T)$  paid to customers. In our framework, we express it as follows:*

$$IRM_g(K_T, L_T) = \delta T \cdot K_T (L_T - g(L_T)). \quad (6)$$

Usually,  $\delta T = 1/4$  corresponds to the usual time interval at which the net interest income is measured in SEC reports. This also corresponds to the time interval between two consecutive ALM committees in most of banking establishments. However, in the following sections, we will consider  $\delta T = 1$  for convenience; rescaling to  $\delta T = 1/4$  is straightforward.

## 4. Hedging Strategies for Interest Rate Margins

### 4.1. Sets of hedging strategies

In this section we define several sets of payoffs each corresponding to a hedging strategy. From now on, we will use the two expressions to designate the same concept.

First, we consider strategies based upon Forward Rate Agreements (FRAs) contracted at initial date. In this case, the quantity to optimize is the amount to be invested in such FRAs. We represent these static strategies by the following set of payoffs:

$$H_{S1} = \{\theta(L_0 - L_T), \theta \in \mathbf{R}\} \quad (7)$$

Let us remark that all claims in  $H_{S1}$  are attainable through some static investments in FRAs and the corresponding price is equal to zero.

Then, we can extend the set of strategies  $H_{S1}$  to a wider set of zero cost investment strategies:

$$H_{S2} = \{\varphi(L_T) \mid \varphi \in \Phi, \forall \mathbf{Q} \in \Pi_{RN}, \mathbf{E}^{\mathbf{Q}}[\varphi(L_T)] = 0\} \quad (8)$$

where  $\Phi$  is the set of functions  $\varphi: \mathbf{R} \rightarrow \mathbf{R}$  such that  $\varphi(L_T)$  is square integrable with respect to  $\mathbf{P}$  and  $\Pi_{RN}$  is the set of risk neutral probability measures. Then,  $H_{S2}$  contains the zero cost European-type options on the final Libor rate. This set of payoffs clearly contains the preceding one, since all payoffs in  $H_{S1}$  are affine functions of  $L_T$ . Actually, any of the payoffs contained in  $H_{S2}$  can be replicated using a dynamic self-financed portfolio of FRAs contracted at each intermediary date  $0 \leq t \leq T$ . In other words, for each  $S \in H_{S2}$ , there exists an admissible<sup>10</sup> strategy  $\theta = (\theta_t)_{0 \leq t \leq T}$  adapted to the filtration generated by the Libor rate,

such that  $S = \int_0^T \theta_t dL_t$ .

Thus, we might think of considering the larger set of self-financed portfolios where the investment process  $\theta = (\theta_t)_{0 \leq t \leq T}$  is adapted to the filtration generated by the Libor rate and allowing for path dependence of the risk profile towards the Libor rate process. Actually, due to the use of the quadratic criterion, the optimal strategy in this larger set belongs to  $H_{S2}$  and thus coincides with  $\varphi^{S2}(L_T)$  defined in subsection 4.3. We give a proof of that in Appendix B. Let us emphasize that, in the latter cases, the investment process  $\theta$  involves the

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<sup>10</sup> From now on, we refer to the notion of admissibility as defined in Gouriéroux, Laurent and Pham (1998) (Definition 2.1) and Pham, Rheinländer and Schweizer (1998) (Section 1.). This definition ensures the closedness of the set  $H_D$ , which allows us to refer to the projection theorem to find optimal hedging strategies.

information carried by the interest rate process and does not deal with that carried by the deposit amount process.

Thus, when using the set  $H_{S_2}$ , the dynamic strategy is myopic regarding the amount of demand deposit at each intermediary date though such information is easily available to any asset and liability manager. Therefore we propose to extend the set  $H_{S_2}$  once further, to a larger set of dynamic self-financed portfolios of FRAs, enabling the manager to adapt his hedging strategy to the evolution of the demand deposit amount. We thus define the following set:

$$H_D = \left\{ V_T(\theta) := \int_0^T \theta_t dL_t, \theta \in \Theta \right\} \quad (9)$$

where  $\Theta$  is the set of admissible strategies<sup>11</sup>, adapted to the filtration generated by  $W_L$  and  $W_K$ . By construction we readily have  $H_{S_1} \subset H_{S_2} \subset H_D$ . These are all closed subspaces of  $L^2(\mathbf{P})$  (see e.g. Delbaen et al. (1997)).

#### 4.2. Mean-Variance Objective

We aim at reducing the interest rate margin's variance using the latter sets of strategies. Thus, for each  $H \in \{H_{S_1}, H_{S_2}, H_D\}$ , we consider the problem:

$$\min_{S \in H} \mathbf{Var}^{\mathbf{P}} [IRM_g(K_T, L_T) - S] \quad (10)$$

This nearly constitutes a problem of orthogonal projection of the margin upon each set of strategies.

Indeed, if we add a return constraint by setting, for each  $H \in \{H_{S_1}, H_{S_2}, H_D\}$  and  $m \in \mathbf{R}$ ,  $H(m) = \{S \in H \mid \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - S] = m\}$ , the latter problem becomes:

$$\min_{S \in H(m)} \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - S]^2 \quad (11)$$

As we mentioned in subsection 3.2, dealing with such a return constraint makes sense due to the risk premium  $\lambda = \frac{\mu_L}{\sigma_L}$  on forward interest rates.

Therefore, the previous problem can be solved through some orthogonal projection of the interest rate margin upon each of the sets of strategies. In particular, we notice that the orthogonal projection on  $H_{S_2}(m)$  of the optimal dynamic portfolio in  $H_D(m)$  leads to the optimal static strategy in  $H_{S_2}(m)$ .

#### 4.3. Optimal Static Strategies

When using only FRAs contracted at initial date (case of  $H_{S_1}$ ), setting a constraint on the return reduces the set of available strategies to a single point. The risk mitigation of the

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<sup>11</sup> See footnote n°10.

interest rate margin in the whole set  $H_{S1}$  is associated with a linear regression of the margin with respect to  $L_T$  and readily yields the quantity

$$\theta^{S1} = \frac{\mathbf{Cov}^{\mathbf{P}}(L_T, \text{IRM}_g(K_T, L_T))}{\mathbf{Var}^{\mathbf{P}}(L_T)} \quad (12)$$

to invest. This basic hedging result heads back for example to Anderson and Danthine (1983).

As for  $H_{S2}$ , we notice that any payoff  $\varphi(L_T)$  has the same expectation under any risk neutral measure. As suggested in Föllmer and Schweizer (1990), we particularly consider the martingale minimal measure defined as follows.

*Definition 4.1.* Consistently with Föllmer and Schweizer (1990), we define the usual minimal martingale measure  $\bar{\mathbf{P}}$  thanks to its Radon-Nikodym density towards the ‘historical’ probability measure  $\mathbf{P}$  by:

$$\frac{d\bar{\mathbf{P}}}{d\mathbf{P}} = \exp(-\lambda^2 T - \lambda W_L(T)) \quad (13)$$

We recall that  $\lambda = \frac{\mu_L}{\sigma_L}$  is the interest rate risk premium involved in investing in long term bonds financed by short term liabilities. Clearly, when  $\lambda = 0$ , there is no difference between the variance minimal and the historical measure.

The minimal martingale measure can be considered as an effective pricing measure for interest rate derivatives. Any payoff  $\varphi(L_T)$  as defined in  $H_{S2}$  can be replicated and the corresponding replication (forward) price is  $\mathbf{E}^{\bar{\mathbf{P}}}[\varphi(L_T)]$ . Moreover, in our framework, the minimal martingale measure  $\bar{\mathbf{P}}$  coincides with the variance minimal measure as defined in Pham, Rheinländer and Schweizer (1998) or Delbaen et al. (1997).

Then, considering the problem with  $H_{S2} = \{\varphi(L_T) \mid \varphi \in \Phi, \mathbf{E}^{\bar{\mathbf{P}}}[\varphi(L_T)] = 0\}$ , we show in Appendix B that the optimal risk profile is then given by:

$$\varphi^{S2}(x) = \mathbf{E}^{\mathbf{P}}[\text{IRM}_g(K_T, L_T) \mid L_T = x] - \mathbf{E}^{\bar{\mathbf{P}}}[\text{IRM}_g(K_T, L_T)] \quad (14)$$

We recall that  $\varphi^{S2}(L_T)$  corresponds to a non constrained optimization, that is the minimization of the variance of the margin without return constraint.

This risk profile involves a non linear regression of the interest rate margin with respect to the Libor rate at date  $T$ . Let us emphasize that on practical grounds, the optimal risk profile can be achieved either through a dynamic replication of  $\varphi^{S2}$  or through a buy-and-hold investment in a European-type option on the interest rate derivatives market.

#### 4.4. Optimal Dynamic Strategies

In this subsection, we deal with the dynamic version of the problem:

$$\min_{S \in H_D} \mathbf{Var}^{\mathbf{P}}[\text{IRM}_g(K_T, L_T) - S] \quad (15)$$

where  $H_D = \{V_T(\theta), \theta \in \Theta\}$ . We recall that  $\Theta$  is the set of admissible strategies that involve the information contained in both the interest rate process and the deposit amount process. Due to the quadratic nature of this problem, its solution fulfils moment conditions which are also detailed in the following theorem.

*Theorem 4.2. Optimal dynamic strategy. The solution  $\theta^{**}$  to  $\min_{\theta \in \Theta} \mathbf{Var}^P [IRM_g(K_T, L_T) - V_T(\theta)]$  is recursively determined by:*

$$\theta_t^{**} = \frac{\partial}{\partial L_t} \mathbf{E}_t^{\bar{P}} [IRM_g(K_T, L_T)] + \frac{\lambda}{\sigma_L L_t} \left[ \mathbf{E}_t^{\bar{P}} [IRM_g(K_T, L_T)] - x^{**} - V_t(\theta^{**}) \right], \quad (16)$$

with  $x^{**} = \mathbf{E}^{\bar{P}} [IRM_g(K_T, L_T)]$ .

Besides, the strategy also verifies the following moment conditions:

$$\forall \theta \in \Theta, \mathbf{E}^P \left[ (IRM_g(K_T, L_T) - x^{**} - V_T(\theta)) \cdot V_T(\theta) \right] = 0. \quad (17)$$

We give a proof of this theorem in Appendix C.

In this theorem, we split the optimal investment strategy into two parts:

- the delta of the interest rate margin under the variance minimal (risk neutral) measure, which acts here as a pricing measure:  $\theta_{\Delta,t}^{**} = \frac{\partial}{\partial L_t} \mathbf{E}_t^{\bar{P}} [IRM_g(K_T, L_T)]$ ;
- some feedback corrective term:  $\theta_{F,t}^{**} = \mathbf{E}_t^{\bar{P}} [IRM_g(K_T, L_T)] - x^{**} - V_t(\theta^{**})$ .

The latter term stands for an investment in the *hedging numéraire* defined in Pham, Rheinländer and Schweizer (1998) and Gouriéroux, Laurent, Pham (1998). The *hedging numéraire* corresponds to some self-financed portfolio, which minimizes its final quadratic dispersion, while aiming at some fixed return of  $(-1)$ . Then, in our framework, the *hedging numéraire* is related to some variance minimization problem in complete market and is only related to the dynamics of the Libor rate process. More precisely, in our case, the *hedging numéraire* at each date is a power function of the Libor rate (see Appendix C).

Let us also point out that, from a mathematical viewpoint, the conditional expectation which appears in the delta term  $\theta_{\Delta,t}^{**}$  stands for the optional projection of the interest rate margin upon the overall filtration<sup>12</sup> at date  $t$ , as defined in Protter (2003).

We also notice that the optimal strategy is linear with respect to the interest rate margin. This implies that, by summing the optimal dynamic strategies for each balance sheet item, our study yields dynamic hedging strategies for the assessment of the risk related to a bank's global net interest income. This widens the perspective initially opened by the study of demand deposit interest rate margins.

The particular case of a linear deposit rate can be solved using Duffie and Richardson's (1991) results<sup>13</sup>. Hopefully, in this linear case, the two approaches lead to the same analytical formulas.

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<sup>12</sup> The overall filtration stands for the filtration related to both  $W_L$  and  $W_K$ .

As stated above, adding a constraint on the expected return  $\mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - V_T(\theta)] = m$  introduces the mean-variance tradeoff. As the following Corollary shows, this slightly modifies the feedback corrective term in the optimal investment strategy.

*Corollary 4.3. Optimal dynamic strategy in mean-variance framework. The solution  $\theta^*(m)$  to  $\min_{\theta \in \Theta} \mathbf{Var}^{\mathbf{P}}[IRM_g(K_T, L_T) - V_T(\theta)]$  u.c.  $\mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - V_T(\theta)] = m$  is recursively determined by:*

$$\theta_t^*(m) = \theta_{\Delta, t}^{**} + \frac{\lambda}{\sigma_L L_t} \left[ \mathbf{E}_t^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)] - x(m) - V_t(\theta^*(m)) \right] \quad (18)$$

$$\text{with } x(m) = x^{**} + \frac{m - x^{**}}{1 - e^{-\lambda^2 T}}.$$

We give a proof of this Corollary in Appendix C.

## 5. Comparison and Robustness of Optimal Static and Dynamic Strategies

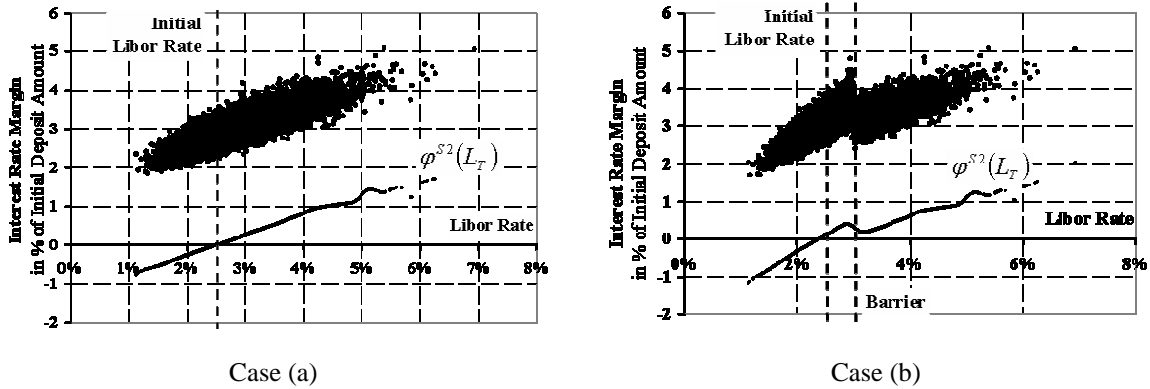
### 5.1. Embedded Optionality in Deposit Accounts

In the following subsections, as an example, we consider parameters corresponding to the Euro Zone as for the deposit amount and the interest rates, that is  $\mu_K = 9.24\%$ ,  $\sigma_K = 6.08\%$ ,  $\mu_L = 5.15\%$ ,  $\sigma_L = 15.42\%$ ,  $\rho = -70.85\%$ . We also deal with a horizon of 2 years ( $T = 2$ ). We set  $K_0 = 100$  as the initial amount of deposits. As stated earlier, for convenience, we report returns and volatilities for a time interval of a year ( $\delta T = 1$ ); rescaling is straightforward.

As an example as well, we consider two different profiles for the deposit rate, corresponding to its modeling in the US (linear:  $g(L_T) = \alpha + \beta L_T$ , case (a)) and in Japan (featuring a barrier:  $g(L_T) = \mathbf{1}_{L_T \geq R}(\alpha + \beta L_T)$ , case (b)). We set, in our example,  $\alpha = -0.50\%$ ,  $\beta = 30\%$  and  $R = 3.00\%$  with  $L_0 = 2.50\%$ . Due to business risk and deposit amount uncertainty, there are several possible levels for the interest rate margin, given the rate  $L_T$ . In Figure 5.1 below, we plotted 20 000 points using a Monte Carlo simulation procedure. Let us also notice that the optimal hedging payoff is nearly linear in case (a), corresponding to a linear deposit rate. We also plotted  $\varphi^{S^2}$  in these two cases (cf. Figure 5.1). We recall that  $\varphi^{S^2}(L_T)$  is the risk profile which best fits the interest rate margin in a quadratic manner (see section 4.3).

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<sup>13</sup> Indeed, due to the affine form of the deposit rate, the interest rate margin is the sum two lognormal random variables. Then, the optimal dynamic strategy is the sum of the optimal dynamic strategies corresponding to each term, which can be determined using Duffie and Richardson's (1991) results.



**Figure 5.1. Interest Rate Margin and Exotic Option Hedge.**

Case (a): linear deposit rate (US case); Case (b): non linear deposit rate (Japanese case). We mentioned the level of  $L_0$  and the barrier in the Japanese case. In both graphs, the upper scatter represents the interest rate margin without hedge (y axis) with respect to  $L_T$  (x axis) and the lower scatter represents the optimal exotic option  $\varphi^{S^2}(L_T)$  in  $H_{S^2}$ . Let us notice that in Case (b),  $\varphi^{S^2}(L_T)$  can be closely reproduced using caps on the rate  $L_T$ .

The previous study shows that the embedded optionality in deposit accounts is influenced by the deposit rate profile, rather than by the business risk. The graphs suggest that the business risk is responsible for some dispersion of the interest rate margin at each level of the terminal Libor rate but has limited influence on the shape of the payoff  $\varphi^{S^2}(L_T)$ , at least given the chosen parameters.

## 5.2. Comparison of Myopic and Dynamic Strategies

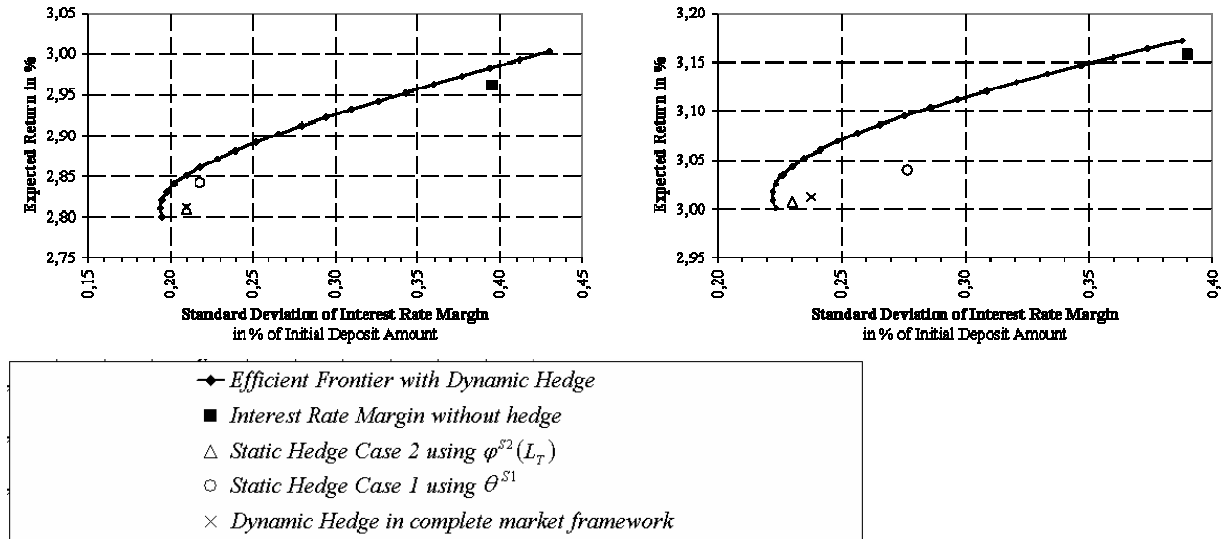
A further issue is the amount of risk reduction involved by dynamically changing the amount of hedging FRAs depending upon the actual amount of deposits. As detailed below, the dependence of the deposit rate with respect to the Libor rate is critical.

Figure 5.4 contains some efficient frontiers related to the problem of dynamic hedging, for two deposit rate profiles – linear and barrier – varying the level of return (see Corollary 4.3.). We also mentioned the points corresponding to the variance minimizing static hedging strategies exhibited in subsection 4.3.

In Jarrow and van Deventer's (1998) framework, the deposit amount is considered as contingent to interest rates only and interest rate margins as interest rate derivatives. In our framework this would consist in neglecting the specific effects of business risk, as suggested in section 3.3 (see (4)). Then, this sets the risk minimization problem into some complete market framework and we computed the usual replicating hedging strategy. Let us emphasize that, in our framework, such a hedging strategy is less risk reducing than  $\varphi^{S^2}(L_T)$ . Indeed, we recall that, for us,  $\varphi^{S^2}(L_T)$  is the most risk reducing payoff among all admissible dynamic strategies adapted with respect to the Libor rate process.

The graphs below show that discrepancies between optimal strategies we defined in section 4 increase as the deposit rate profile becomes more exotic. For example we notice this phenomenon as for the complete market related hedging strategy, comparing to  $\varphi^{S^2}(L_T)$ . Indeed, the complete market hedging strategy performs as well as  $\varphi^{S^2}(L_T)$  when the deposit

rate is affine, but we notice a slight risk shift between them when the deposit rate features a barrier. Actually, this shift is really small, showing some good performances of the complete market approach. However, the linear approximation, in  $H_{S1}$ , of the interest rate margin dramatically lacks precision when the deposit rate goes far from linearity.



**Figure 5.2. Efficient Frontiers and Several Hedging Strategies.** (left: linear deposit rate (US case) – right: non linear deposit rate (Japanese case).

The latter study suggests that fitting the interest rate margin using a function of the terminal Libor rate constitutes a reliable strategy, since it seems to remain close to the efficient frontier whatever the specification of the deposit rate. This is an interesting feature for a strategy, since banks experience tough difficulty identifying the interest rate option within demand deposit interest rate margins.

Comparing the method based upon  $\varphi^{S2}(L_T)$  and the amount  $\theta^{S1}$  of FRAs at initial date given by equation (12), we notice that the discrepancy between their performances increases as the profile for the deposit rate goes from affine to discontinuous. Indeed, in Table 5.5 below, we notice that at the same level of return, for affine and zero deposit rates, there is no additional risk alleviation using  $\varphi^{S2}(L_T)$  (*Static Hedge Case 2*) rather than investing the amount  $\theta^{S1}$  in FRAs (*Static Hedge Case 1*). However, when the deposit rate features a more exotic profile (barrier), we notice that  $\varphi^{S2}(L_T)$  slightly over-performs the linear hedge.

Hedging Methods		No Deposit Rate	Linear Deposit Rate (US case)	Barrier Deposit Rate (Japanese case)
Static Hedge Case 1 Hedging with FRAs contracted at $t=0$	Return	3.12	2.84	3.04
	St. Dev.	0.24	0.22	0.28
	Sharpe Ratio	2.59	1.57	1.95
Static Hedge Case 2 Hedging with the optimal risk profile on the terminal Libor Rate $L_T$	Return	3.12	2.84	3.04
	St. Dev.	0.24	0.22	0.24
	Sharpe Ratio	2.60	1.54	2.25

**Table 5.3. Comparing Static Hedging Strategies.**

‘Static Hedge Case 1’ corresponds to the case of  $H_{S1}$  and ‘Static Hedge Case 2’, to the case of  $H_{S2}$  (cf. subsection 4.1.). Results are given for an initial deposit amount of  $K_0 = 100$ . We also mentioned the Sharpe ratios with respect to some riskless investment in the forward rate  $L_0$ .

Comparing to static strategies using functions of  $L_T$  (contained in  $H_{S2}$ ), the information contained in the deposit amount process is taken into account in the optimal dynamic strategy (in  $H_D$ ). At first sight, this should lead to a better alleviation of the specific risk embedded in demand deposit amount.

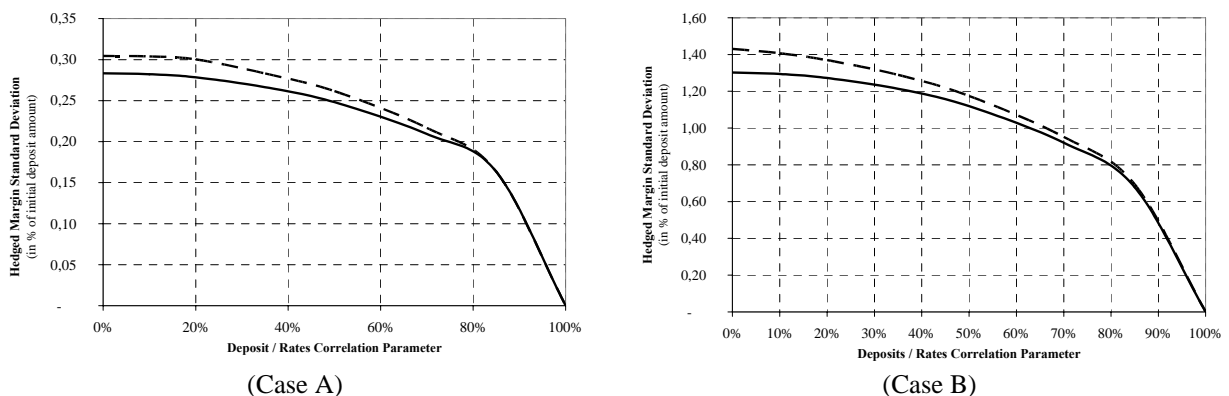
Indeed, in Table 5.6 below, we notice that the shift between the minimal standard deviation for each kind of strategy increases from 8.6% to 11.4% as the correlation between the deposit amount and interest rates gets smaller. In other words, when the relative influence of the business risk increases – comparing with the market risk – the dynamic strategy increasingly over-performs the static strategy. As expected, the risk profile  $\varphi^{S2}(L_T)$ , thus the complete market hedging strategy, feature imperfections as for the assessment of the specific risk carried in deposits.

Correlation parameter $\rho$	-100%	-90%	-65%	-30%	-10%	0%
<b>Optimal Dynamic Hedge</b>						
Assessing jointly the information contained in the deposit amount and the market rate	0.000	0.124	0.216	0.272	0.285	0.287
<b>Static Hedge Case 2</b>						
Hedging with the optimal risk profile on the terminal Libor rate	0.000	0.134	0.236	0.301	0.317	0.320
<b>Relative Difference</b>	N/A	8.6%	9.4%	10.5%	11.1%	11.4%

**Table 5.4. Minimal Standard Deviation for Dynamic Hedging Strategies for Various Levels of the Correlation Parameter.**

Standard deviations are given in % of the initial deposit amount. The ‘Optimal Dynamic Hedge’, corresponds to the left endpoint of the efficient frontier (see above), thus referring to the case of  $H_D$ . The ‘Static Hedge Case 2’ corresponds to the case of  $H_{S2}$ . The ‘Relative Difference’ assesses the relative shift of line 2 over line 1.

When  $\rho = -100\%$ , there is no specific deposit risk to assess, thus implying no difference between the two hedging methods. However, when  $|\rho|$  decreases, some incompleteness appears and using myopic hedging strategies as for the deposit amount process yields differences in the risk assessment in interest rate margins (from 8.4% to 11.6%). The results are summarized in Figure 5.5 below.



**Figure 5.5. Minimal Standard Deviation with respect to Correlation Parameter  $\rho$  (left:  $\sigma_K = 6.08\%$  , right:  $\sigma_K = 30\%$  ).**

We represented the standard deviation of the hedged margin at minimal variance point in function of the correlation parameter  $\rho$  , for a barrier customer rate, and for two levels of deposit volatility. The dotted line corresponds to the use of the risk profile  $\varphi^{S_2}(L_T)$  and the continuous line, to the use of the optimal dynamic hedging strategy in  $H_D$  .

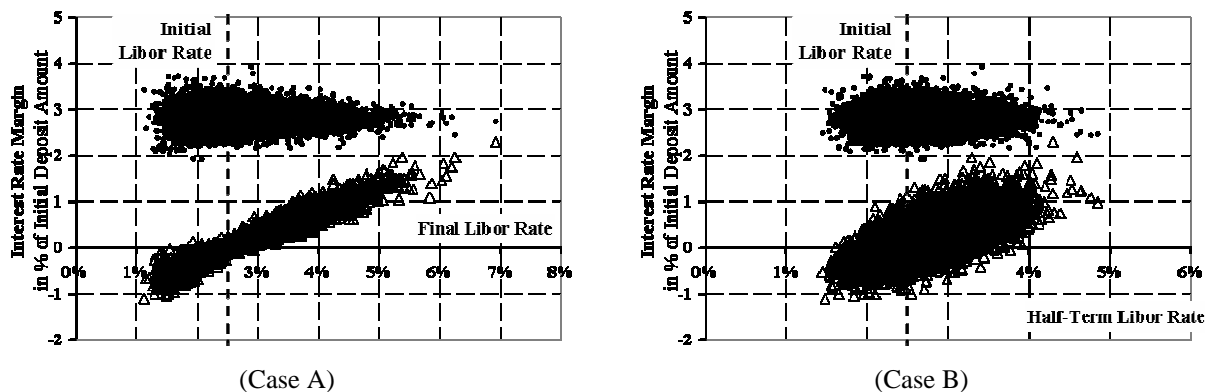
### 5.3. Checking Optimality of Dynamic Strategies

In this subsection, we check the independence of the dynamically hedged margin (in the case of  $H_D$  (see subsection 4.4)), towards self-financed strategies on the Libor rate process, at minimum variance point<sup>14</sup>. Indeed, Theorem 4.2 states that the strategy  $\theta^{**}$  minimizing the variance fulfils the usual first order conditions, that is  $\text{Cov}^P(IRM_g(K_T, L_T) - V_T(\theta^{**}), V_T(\theta)) = 0$  for any admissible strategy  $\theta \in \Theta$ . This reminds us about the condition fulfilled by the GMM estimator, thus the related tests of specification may be applied here, using simulations (cf. Appendix A.2 in Campbell, Lo and MacKinlay (1997) for example). This can be viewed as a numerical check of the optimality of hedging strategies.

For example we used the Fisher test for zero correlation, in two cases<sup>15</sup>:

- between  $IRM_g(K_T, L_T) - V_T(\theta^{**})$  and  $V_T(\theta) = L_T - L_0$  (case A);
- between  $IRM_g(K_T, L_T) - V_T(\theta^{**})$  and  $V_T(\theta) = L_{T/2} - L_0$  (case B),

using 20 000 simulations and the correlations satisfy the Fisher test at 1% and 5% confidence levels. Then, hopefully, the optimal strategy fulfils the first order condition in the latter two cases. The set of parameters is the same as in subsection 5.1. The graphs in Figure 5.5 summarize the independence of the hedged margin with respect to the final Libor rate and the half-term Libor rate.



**Figure 5.5. Hedged Interest Rate Margin and Optimal Hedging Portfolio with respect to Final Libor Rate (left) and Half-Time Libor Rate (right).**

We represented the optimally hedged interest rate margin (in dots) and the associated hedging portfolio (in empty triangles) in function of the final Libor rate (Case A – left) and the half-term Libor rate (Case B – right).

<sup>14</sup> We recall that by “minimum variance point”, we mean the optimum in the problem  $\min_{S \in H_D} \text{Var}^P [IRM_g(K_T, L_T) - S]$ .

<sup>15</sup> The first case corresponds to  $\theta_t = 1$  for any  $0 \leq t \leq T$  ; the second corresponds to  $\theta_t = 1$  for  $0 \leq t \leq T/2$  and  $\theta_t = 0$  for  $T/2 \leq t \leq T$  .

#### 5.4. Robustness towards Risk Criterion

Banks' internal risk measurement procedures often involve Value-at-Risk and Expected Shortfall computations (cf. definitions in Acerbi and Tasche (2002) for example). In Tables 5.6a and 5.6b, we propose to compare the performances of different hedging strategies for these criteria. We choose a 99.95% threshold for the VaR and 99.5% for the ES.

<i>Barrier Deposit Rate</i>	Expected Return	Standard Deviation		ES (99.5%)		VaR (99.95%)	
		Level	Risk Reduction	Level	Risk Reduction	Level	Risk Reduction
<b>Unhedged Margin</b>	3.16	0.39		-2.02		-1.90	
<b>Static Hedge Case 1</b>	3.04	0.28	-0.11	-2.34	-0.32	-2.26	-0.36
<b>Static Hedge Case 2</b>	3.01	0.23	-0.16	-2.26	-0.24	-2.04	-0.14
<b>Without 'Specific' Risk</b>	3.01	0.24	-0.15	-2.35	-0.33	-2.25	-0.35
<b>Optimal Dynamic Hedge</b>	3.01	0.22	-0.17	-2.38	-0.36	-2.29	-0.39

**Table 5.6a. Risk Measurement for Various Hedging Methods – Barrier Deposit Rate.**

We computed in the upper line the risk measures of the unhedged margin and mentioned in the following lines the level and the gain when choosing each kind of strategy. *Static Hedge Case 1* stands for the hedging strategy using only FRAs at terminal date (case of  $H_{S1}$ ); *Static Hedge Case 2* stands for the fitting of the interest rate margin using a function of the terminal Libor rate (case of  $H_{S2}$ ). *Without 'Specific Risk'* corresponds to the framework in which there is no deposit 'specific' risk, as described in Section 3.3 and Equation (4).

The negative values for the ES and the VaR in the upper line are due to the fact that the margin at final date is mostly positive<sup>16</sup>. Thus, the VaR at 99.95% of the interest rate margin is (-1.90) for an initial deposit amount of 100. We see, in Table 5.6a above, that using the optimal dynamic hedging strategy makes the risk decrease by 0.39 to (-2.29), thus constituting a better risk reduction than other strategies, in this VaR framework. The same actually holds for the ES and the standard deviation.

<i>No Deposit Rate</i>	Expected Return	Standard Deviation		ES (99.5%)		VaR (99.95%)	
		Level	Risk Reduction	Level	Risk Reduction	Level	Risk Reduction
<b>Unhedged Margin</b>	3.35	0.58		-2.02		-1.90	
<b>Static Hedge Case 1</b>	3.12	0.24	-0.34	-2.42	-0.40	-2.29	-0.39
<b>Static Hedge Case 2</b>	3.11	0.24	-0.34	-2.35	-0.33	-2.17	-0.27
<b>Without 'Specific' Risk</b>	3.11	0.24	-0.34	-2.35	-0.33	-2.25	-0.35
<b>Optimal Dynamic Hedge</b>	3.11	0.22	-0.36	-2.47	-0.45	-2.36	-0.46

**Table 5.6b. Risk Measurement for Various Hedging Methods – No Deposit Rate.**

Same caption as above.

In general, the risk reduction implied by the optimal dynamic strategy is almost always more significant: even when there is no customer rate (Table 5.6b), it goes to 0.45 in ES and 0.46 in VaR. This shows some robustness of the optimal dynamic strategy also with respect to the choice of the risk criterion. Moreover, this somehow makes us confident with the mean-variance optimization framework, more tractable than some mean-VaR or mean-ES framework.

## 6. Conclusions

<sup>16</sup> Here, consistently with Acerbi and Tasche (2002), we define the VaR at level 99.95% as the opposite of the 0.05% quantile of the distribution. We use the same convention for the Expected Shortfall.

In this article we dealt with the mitigation of the risk contained in interest rate margins. We assume the demand deposit amount to carry some source of risk called ‘business risk’, orthogonal to market risk. Thus we deal with mean-variance hedging of the margins in some incomplete market framework; thanks to Duffie and Richardson’s (1991) results and the theory of the hedging numéraire, we derive explicit dynamic hedging strategies. There are various ways to model the demand deposit amount and rates, but the method we developed in this article can cope with a wide range of them.

Thus, we compared these dynamic strategies with some static strategies. We show that identifying the interest rate-related optionality in interest rate margins is a quite satisfactory alternative to dynamic hedging strategies. Moreover, both this method and the use of dynamic strategies lead to quite robust results, with respect to the margin’s profile specification.

However, the use of dynamic strategies also better deals with the specific risk embedded in demand deposits. Moreover, we show that they also exhibit some robustness with respect to the risk criterion. This is a positive conclusion for the use of mean-variance optimization and the related dynamic hedging strategies, since they display good results with respect to other risk measures.

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## Appendix A: List of US Banks used in interest rate margins analysis<sup>17</sup>

Bank Name	City and State	Number of Branches	Asset Size (in millions of dollars)	Net income (year2005) (in millions of dollars)
Webster Bank, National Association	WATERBURY, CT	179	16 622	468
Associated Bank, National Association	GREEN BAY, WI	310	20 312	553
Colonial Bank, National Association	MONTGOMERY, AL	321	23 325	567
Compass Bank	BIRMINGHAM, AL	415	36 914	780
TD BankNorth, National Association	PORTLAND, ME	587	42 368	927
Fifth Third Bank	CINCINNATI, OH	408	53 249	3 048
M&I Marshall and Ilsley Bank	MILWAUKEE, WI	290	49 334	1 132
Union Bank of California, National Association	SAN FRANCISCO, CA	320	52 743	1 624
The Huntington National Bank	COLUMBUS, OH	710	54 186	911
Manufacturers and Traders Trust Company	BUFFALO, NY	637	56 713	1 735
Bank of the West	SAN FRANCISCO, CA	663	56 963	1 352
Comerica Bank	DETROIT, MI	394	58 088	1 810
Capital One, National Association	MCLEAN, VA	695	95 204	3 003
PNC Bank, National Association	PITTSBURGH, PA	1 055	117 232	1 969
Branch Banking and Trust Company	WINSTON SALEM, NC	1 473	120 906	3 348
National City Bank	CLEVELAND, OH	1 363	135 636	4 433
HSBC Bank USA, National Association	WILMINGTON, DE	453	166 101	7 802
SunTrust Bank	ATLANTA, GA	1 711	174 962	3 685
Commerce Bank, National Association	PHILADELPHIA, PA	405	43 053	497
First Tennessee Bank, National Association	MEMPHIS, TN	262	38 178	856

## Appendix B: Optimal Strategy in $H_{S_2}$ .

First we look for the solution  $\varphi_m^*$  to the constrained problem:

$$\begin{aligned} \min_{\varphi \in \Phi} \mathbf{Var}^{\mathbf{P}} [IRM_g(K_T, L_T) - \varphi(L_T)] \\ \text{u.c. } \mathbf{E}^{\bar{\mathbf{P}}}[\varphi(L_T)] = 0 \text{ and } \mathbf{E}^{\mathbf{P}}[IRM_g(K_T, L_T) - \varphi(L_T)] = m \end{aligned} \quad (\text{B1})$$

and then we deal with:

$$\min_m \mathbf{Var}^{\mathbf{P}} [IRM_g(K_T, L_T) - \varphi_m^*(L_T)] \quad (\text{B2})$$

We recall that  $\Phi$  is the set of functions  $\varphi: \mathbf{R} \rightarrow \mathbf{R}$  such that  $\varphi(L_T)$  is square integrable with respect to  $\mathbf{P}$ .

Due to its convexity, the constrained problem (B1) is equivalent to:

$$\begin{aligned} \min_{f, \lambda, \mu} \Gamma(f, \lambda, \mu) = \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - f(L_T)]^2 + 2\lambda \mathbf{E}^{\mathbf{P}} \left[ \frac{d\bar{\mathbf{P}}}{d\mathbf{P}} f(L_T) \right] \\ + 2\mu \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - f(L_T) - m] \end{aligned}$$

which yields:

$$\varphi_m^*(L_T) = \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) | L_T] - \lambda \frac{d\bar{\mathbf{P}}}{d\mathbf{P}} + \mu \text{ with } \begin{cases} -\lambda \mathbf{E}^{\mathbf{P}} \left[ \frac{d\bar{\mathbf{P}}}{d\mathbf{P}} \right]^2 + \mu = -\mathbf{E}^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)] \\ -\lambda + \mu = -m \end{cases}$$

The optimal  $m^*$  in (B2) is then given by:

$$m^* = \mathbf{E}^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)],$$

yielding  $\varphi_{S_2}^*(L_T) = \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) | L_T] - \mathbf{E}^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)]$  as expected.

<sup>17</sup> Source FFIEC ([www.ffiec.org](http://www.ffiec.org)) and SEC ([www.secinfo.com](http://www.secinfo.com)).

**Dealing with path-dependence towards the forward Libor rate process**

We consider the set  $H_{S_2}^* = \left\{ \int_0^T \theta_s dL_s, \theta \in \Theta^L \right\}$  where  $\Theta^L$  is the set of  $\theta \in \Theta$  which are

adapted to the filtration related to the forward Libor rate process. Here we prove that

$$\min_{S \in H_{S_2}^*} \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - S \right]^2 = \min_{S \in H_{S_2}} \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - S \right]^2.$$

Given any  $S \in H_{S_2}^*$  we have:

$$\begin{aligned} \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - S \right]^2 &= \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) \middle| F^L \right] \right]^2 \\ &\quad + \mathbf{E}^P \left[ \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) \middle| F^L \right] - S \right]^2 \end{aligned}$$

since  $S$  is  $F^L$ -measurable.

In our framework,  $\mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) \middle| F^L \right]$  can be set under the form  $\varphi(L_T)$  for some  $\varphi \in \Phi$ .

Therefore, the minimum in  $\min_{S \in H_{S_2}^*} \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - S \right]^2$  is attained in  $H_{S_2}$ .

**Appendix C: Proof of Theorem 4.2.**

We provide here a proof for Theorem 4.2 and Corollary 4.3.

First, we recall the notion of *hedging numéraire*, defined in Gouriéroux, Laurent and Pham (1998) and Pham, Rheinländer and Schweizer (1998).

*Definition C1. Hedging Numéraire.* The hedging numéraire is the value process of the dynamic portfolio solving

$$\min_{\theta \in \Theta} \mathbf{E}^P \left[ 1 + V_T(\theta) \right]^2.$$

In our framework, the *hedging numéraire* is a power function of the forward Libor rate and

verifies  $\frac{dN_t}{N_t} = -\frac{\lambda}{\sigma_L} \frac{dL_t}{L_t}$  at each date  $t$ .

Using Lemma 3 in Duffie and Richardson (1991), we readily establish that the problem  $\min_{\theta \in \Theta} \mathbf{Var}^P \left[ \text{IRM}_g(K_T, L_T) - V_T(0, \theta) \right]$  yields the same optimal strategy  $\theta^{**}$  as the problem

$$\min_{\substack{x \in \mathbf{R} \\ \theta \in \Theta}} \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - x - V_T(\theta) \right]^2.$$

Then, Propositions 3.2 and 5.1 in Gouriéroux, Laurent and Pham (1998) show that solving the problem  $\min_{\theta \in \Theta} \mathbf{E}^P \left[ \text{IRM}_g(K_T, L_T) - (x + V_T(\theta)) \right]^2$  for some  $x \in \mathbf{R}$  yields the following solution:

$$\theta_t^*(x) = \left( \frac{\partial \bar{L}_t}{\partial L_t} \right)^{-1} \left( \frac{\partial}{\partial L_t} \mathbf{E}_t^{\bar{\mathbf{P}}_{HN}} \left[ \frac{\text{IRM}_g(K_T, L_T)}{N_T} \right] \right) \left( 1 + \frac{\lambda}{\sigma_L} \right) - \frac{\lambda}{\sigma_L L_t} (x + V_t(\theta^*)), \quad (C1)$$

where  $N = (N_t)_{0 \leq t \leq T}$  is the value process of the *hedging numéraire* (see above) and  $\bar{\mathbf{P}}_{HN}$ , the probability measure equivalent to the variance minimal measure  $\bar{\mathbf{P}}$  (see our Definition 4.1)

and defined by  $\frac{d\bar{\mathbf{P}}(a^{HN})}{d\bar{\mathbf{P}}} = N_T$  (see Proposition 3.1 in Gouriéroux et al. (1998)). Finally,

$\bar{L} = \frac{L}{N}$  corresponds to the forward Libor process in *hedging numéraire* units. Then, the optimal strategy becomes:

$$\theta_t^*(x) = \frac{\partial}{\partial L_t} \mathbf{E}_t^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)] - \frac{\lambda}{\sigma_L L_t} \left[ \mathbf{E}_t^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)] - x - V_t(\theta^*) \right].$$

Then the optimal  $x$  in problem  $\min_{\substack{x \in \mathbf{R} \\ \theta \in \Theta}} \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - x - V_T(\theta)]^2$ , called the *approximation price* in Gouriéroux et al.'s (1998) Theorem 5.2, is given by  $x^{**} = \mathbf{E}^{\bar{\mathbf{P}}} [IRM_g(K_T, L_T)]$ .

The justification for moment conditions is exactly the same as in Duffie and Richardson's (1991) Lemma 1 and Pham, Rheinländer and Schweizer's (1998) Theorem 7.

This achieves the proof of Theorem 4.2.

### ***Proof of Corollary 4.3***

We recall that Propositions 3.2 and 5.1 in Gouriéroux et al. (1998) state that solving the problem  $\min_{\theta \in \Theta} \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - x - V_T(\theta)]^2$  for some  $x \in \mathbf{R}$  yields:

$$\theta_t^*(x) = \left( \frac{\partial \bar{L}_t}{\partial L_t} \right)^{-1} \left( \frac{\partial}{\partial L_t} \mathbf{E}_t^{\bar{\mathbf{P}}_{HN}} \left[ \frac{IRM_g(K_T, L_T)}{N_T} \right] \right) \left( 1 + \frac{\lambda}{\sigma_L} \right) - \frac{\lambda}{\sigma_L L_t} (x + V_t(\theta^*)),$$

with the same notations as above.

Then we have:

$$\mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - V_T(\theta^*(x))] - \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - V_T(\theta^{**})] = (1 - \mathbf{E}^{\mathbf{P}}[N_T])(x - x^{**}) \quad (\text{C2})$$

with  $\mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - V_T(\theta^{**})] = x^{**}$  and  $\mathbf{E}^{\mathbf{P}}[N_T] = e^{-\lambda^2 T}$ .

Thanks to Lemma 4.3 in Duffie and Richardson (1991), we state that  $\theta_t^*(x)$  solves  $\min_{\theta \in \Theta(m)} \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - V_T(\theta)]^2$  where  $m = \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - V_T(\theta^*(x))]$ . In other words, according to our notations,  $\theta^*(x)$  coincides with  $\theta^*(m)$  for  $m = \mathbf{E}^{\mathbf{P}} [IRM_g(K_T, L_T) - V_T(\theta^*(x))]$ . Then, from Theorem 5.1 in Gouriéroux et al. (1998), we have  $\frac{\partial V_t(\theta_t^*(x))}{\partial x} = N_t$  and this yields:

$$\frac{\partial \theta_t^*(x)}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial V_t(\theta_t^*(x))}{\partial L_t} \right) = \frac{\partial}{\partial L_t} \left( \frac{\partial V_t(\theta_t^*(x))}{\partial x} \right) = \frac{\partial N_t}{\partial L_t} = -\frac{\lambda}{\sigma_L L_t} N_t$$

Integrating the latter equation between  $x^{**}$  and  $x$  and using (C2), we obtain:

$$\theta_t^*(m) = \theta_t^{**} - \frac{\lambda}{\sigma_L L_t} N_t (x - x^{**}) = \theta_t^{**} - \frac{\lambda}{\sigma_L L_t} \frac{N_t}{(1 - \mathbf{E}^{\mathbf{P}}[N_T])} (m - x^{**}).$$

This achieves the proof of Corollary 4.3.