THE USE OF THRESHOLD COPULAS FOR DEFINING CONTAGION AMONG FINANCIAL MARKETS

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ABSTRACT. We propose a method for defining and investigating contagion between two financial markets X and Y by using the information about their dependence, as described by their copula $C_{(X,Y)}$.
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1. INTRODUCTION

In the last decades, several financial crises stressed the fact that markets tend to be more dependent during a crisis than they are during calmer periods. This situation is usually referred as *contagion*, which has recently attracted the attention of several theoretists and practitioners working on finance due to its dramatical effects. In the literature, several methodologies have been proposed for checking whether contagion is occurred. The major part of them aims at searching for the change in structure of the underlying distribution governing the behaviour of return series. See, for example, [15, 1, 22, 10, 24] and the references therein.

Another approach has been recently provided in [2, 3, 4]. At an intuitive level, they say that there is contagion from market X to market Y if there is more dependence between X and Y when X is doing badly than when X exhibits typical performance, that is, if there is more dependence at the loss distribution of X than at its center. These changes of dependence are, in their approach, measured by means of some *local correlation coefficient*. As the same authors said (compare with [2]), this approach "does not require any definition of crisis and normal periods and it is not temporal in nature". It is usually referred to as *spatial* contagion in order to reflect the fact that the strength of the dependence between X and Y is measured at different points of their distributions.

In this paper, following the ideas and the motivations by Bradley and Taqqu, we aim at defining a related notion of contagion among financial markets by describing their dependence not by means of local correlation, but by using *copulas*. As well known from people working in applied probability and finance, copulas are functions that describe the dependence among random variables and, at the same time, avoid some pitfalls and unnecessary simplifications given by correlation coefficients [14].

In section 2, we present some results about threshold copulas, which will be useful in the rest of the paper. Threshold copulas are essentially the objects that allow to express the dependence for conditional d.f.'s, and,

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hence, describe the different behaviour of some joint distribution functions in some specific regions. Our definition of contagion are then explained in section 3, with a number of examples showing possible pitfalls and fallacies in the interpretation of contagion. An empirical illustration is hence provided in section 4.

2. Description of the dependence for tail and central events

Let X and Y be two continuous random variables (=r.v.'s) defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that X and Y are continuous with joint distribution function (=d.f.) F and (univariate) marginal d.f.'s F_X and F_Y. For our purpose, typically, X and Y represent the returns of two financial markets.

It is well known that the dependence among X and Y is fully captured by a function $C := C_{(X,Y)}$, called *copula*, which is the restriction to $[0,1]^2$ of a bivariate d.f. whose univariate margins are uniformly distributed on $[0,1]$. In fact, thanks to *Sklar's theorem* [26], we have that any distribution function F can be represented in the form

$$F(x,y) = C(F_X(x), F_Y(y)) \qquad \text{for all } (x,y) \in \mathbb{R}^2. \tag{1}$$

(Note that this representation is unique only when F is continuous). Roughly speaking, copulas allow to split the description of the joint behaviour between X and Y in two parts: the description of the marginal d.f.'s and the description of the dependence structure, fully captured by their copula C. Examples of copulas are: the copula M, given by $M(u,v) = \min(u,v)$, which describes the *comonotone dependence*; the copula W, given by $W(u,v) = \max(u+v-1, 0)$, which describes the *countermonotone dependence*; the copula Π, given by $\Pi(u,v) = uv$, which describes the *independence* When some d.f. F is known, the related copula can be just obtained by means of the formula

$$C(u,v) = F(F_X^{-1}(u), F_Y^{-1}(v)) \qquad \text{for all } (u,v) \in [0,1]^2, \tag{2}$$

where, for any univariate d.f. F, $F^{-1}(u) := \sup\{x \in \mathbb{R} : F(x) \geq u\}$ denotes the *quantile inverse* of F.

Relevant applications of copulas in finance are given, for example, in [7, 20, 19]. A theoretical introduction on this topic can be, instead, found in [16, 9, 21].

Nowadays, several copula–based models have been proposed for describing the dependence among financial markets, especially when one tries to quantify the probability that big losses (respectively, big gains) occur at the same time [20]. A recent and powerful tool for investigating the dependence structure in the tails of the joint d.f. is to consider a conditional model and, then, to investigate the related copula, also called *conditional copula* or *threshold copula* [17, 18, 5, 6, 11]. In the following, we aim at presenting this concept in some general way that will fit better with the purpose of our investigation.

Let (X,Y) be a random pair with continuous joint d.f. F. Let $\mathcal{A}$ be a (convex) subset of $\mathbb{R}^2$ such that $\mathbb{P}\{(X,Y) \in \mathcal{A}\} > 0$. We denote by $F_{(X,Y)|\mathcal{A}}$, or simply $F_{\mathcal{A}}$, the conditional d.f. of (X,Y) given the event $\{\omega \in$

$\Omega : (X(\omega), Y(\omega)) \in \mathcal{A}\}$, defined, for all $(x, y) \in \mathcal{A}$, by

$$F_{\mathcal{A}}(x, y) := \mathbb{P}\left(X \leq x, Y \leq y \mid (X, Y) \in \mathcal{A}\right). \tag{3}$$

In particular, we are interested on the situations where $\mathcal{A}$ assumes some specific forms. The first cases of interest appear when $\mathcal{A}$ is a *tail set* of one of the following types:

- $\mathcal{A}_L = \left]-\infty, q_\alpha(X)\right] \times \mathbb{R}$,
- $\mathcal{A}_R = \left[q_{1-\alpha}(X), +\infty\right[\times \mathbb{R}$,
- $\mathcal{A}_D = \mathbb{R} \times \left]-\infty, q_\alpha(Y)\right]$,
- $\mathcal{A}_U = \mathbb{R} \times \left[q_{1-\alpha}(Y), +\infty\right[$,
- $\mathcal{A}_{LD} = \left]-\infty, q_\alpha(X)\right] \times \left]-\infty, q_\alpha(Y)\right]$,
- $\mathcal{A}_{RU} = \left[q_{1-\alpha}(X), +\infty\right[\times \left[q_{1-\alpha}(Y), +\infty\right[$,

where, for each r.v. Z, $q_\alpha(Z)$ denotes the α–quantile of Z, $\alpha \in \left]0, \frac{1}{2}\right[$. These situations appear when we want to investigate the joint behaviour of (X, Y) supposed that X and/or Y fall under or over a given threshold. In typical situations where X and Y are returns of two markets, we are then mainly interested in determining the joint behaviour of (X, Y) supposing that X and/or Y are having big losses (bearish markets) or big gains (bullish markets).

Second cases appear when $\mathcal{A}$ is a *central set*, like:

- $\mathcal{A}_V = [q_\beta(X), q_{1-\beta}(X)] \times \mathbb{R}$,
- $\mathcal{A}_H = \mathbb{R} \times [q_\beta(Y), q_{1-\beta}(Y)]$,
- $\mathcal{A}_M = [q_\beta(X), q_{1-\beta}(X)] \times [q_\beta(Y), q_{1-\beta}(Y)]$,

where $\beta \in \left]0, \frac{1}{2}\right[$. When X and Y are returns of two markets and $\mathcal{A}$ is a central set, $F_{(X,Y)|\mathcal{A}}$ describes the behaviour of (X, Y) when X and/or Y have no critical performances.

For our purpose, hence, the set $\mathcal{A}$ always corresponds to a rectangle of $\mathbb{R}^2$ (eventually with vertices at infinity). In this case, it is easier to compute the joint d.f. $F_{\mathcal{A}}$ and its related (threshold) copula. Note that all previously introduced (conditional) d.f.'s are continuous and, thus, each of them can be associated with a unique copula. For the next, we recall that, given a bivariate d.f. F and rectangle $[a_1, a_2] \times [b_1, b_2] \subseteq \mathbb{R}^2$, the *volume* of F is defined by

$$V_F([a_1, a_2] \times [b_1, b_2]) = F(a_1, b_1) + F(a_2, b_2) - F(a_1, b_2) - F(a_2, b_1).$$

Proposition 1. *Let $\mathcal{A} := [a_1, a_2] \times [b_1, b_2]$ be a rectangle of $\mathbb{R}^2$, with $-\infty \leq a_1 < a_2 \leq +\infty$ and $-\infty \leq b_1 < b_2 \leq +\infty$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous joint d.f. with marginals F_X and F_Y and copula C. Then, for all $(x, y) \in \mathcal{A}$,*

$$F_{(X,Y)|\mathcal{A}}(x, y) = \frac{V_C([F_X(a_1), F_X(x)] \times [F_Y(b_1), F_Y(y)])}{V_C([F_X(a_1), F_X(a_2)] \times [F_Y(b_1), F_Y(b_2)])}. \tag{4}$$

Proof. Given $\mathcal{A} := [a_1, a_2] \times [b_1, b_2]$, we have that

$$F_{\mathcal{A}}(x, y) = \mathbb{P}(X \leq x, Y \leq Y \mid (X, Y) \in \mathcal{A}) = \frac{V_F([a_1, x] \times [b_1, y])}{V_F([a_1, a_2] \times [b_1, b_2])}.$$

The assertion follows directly by using the fact that, for every $(x, y) \in \mathbb{R}^2$, $F(x, y) = C(F_X(x), F_Y(y))$. □

Now, the copula $C_{(X,Y)|\mathcal{A}}$ of $F_{(X,Y)|\mathcal{A}}$ can be obtained by means of (2). To this end, observe that the marginal d.f.'s of $F_{(X,Y)|\mathcal{A}}$ are given, for all $(x,y)\in\mathcal{A}:=[a_1,a_2]\times[b_1,b_2]$, by

$$F_{X|\mathcal{A}}(x) := F_{\mathcal{A}}(x,+\infty)=\frac{V_C([F_X(a_1),F_X(x)]\times[F_Y(b_1),F_Y(b_2)])}{V_C([F_X(a_1),F_X(a_2)]\times[F_Y(b_1),F_Y(b_2)])}, \tag{5}$$

$$F_{Y|\mathcal{A}}(y) := F_{\mathcal{A}}(+\infty,y)=\frac{V_C([F_X(a_1),F_X(a_2)]\times[F_Y(b_1),F_Y(y)])}{V_C([F_X(a_1),F_X(a_2)]\times[F_Y(b_1),F_Y(b_2)])}. \tag{6}$$

Proposition 2. *Let $A:=[a_1,a_2]\times[b_1,b_2]$ be included into $\mathbb{R}^2$, with $-\infty\le a_1<a_2\le+\infty$ and $-\infty\le b_1<b_2\le+\infty$. Let $F:\mathbb{R}^2\to\mathbb{R}$ be a continuous joint d.f. with marginals F_X and F_Y and copula C. Then, for all $(u,v)\in[0,1]$, the copula $C_{\mathcal{A}}:=C_{(X,Y)|\mathcal{A}}$ of $F_{(X,Y)|\mathcal{A}}$ is given by*

$$C_{\mathcal{A}}(u,v)=\frac{V_C([F_X(a_1),F_X(F^{-1}_{X|\mathcal{A}}(u))]\times[F_Y(b_1),F_Y(F^{-1}_{Y|\mathcal{A}}(v))])}{V_C([F_X(a_1),F_X(a_2)]\times[F_Y(b_1),F_Y(b_2)])}. \tag{7}$$

Example 1. Let us consider the d.f. $F_{(X,Y)|\mathcal{A}_L}$, which describes the joint behaviour of a random pair (X,Y) with joint d.f. F when the r.v. X falls under its quantile $q_\alpha(X)$. This d.f. is given, for every $(x,y)\in\,]-\infty,q_\alpha(X)]\times\mathbb{R}$, by

$$F_{(X,Y)|\mathcal{A}_L}(x,y)=\mathbb{P}(X\le x,Y\le y\mid X\le q_\alpha(X))=\frac{C(F_X(x),F_Y(y))}{\alpha},$$

where C is the copula of F. The marginal d.f.'s of $F_{(X,Y)|\mathcal{A}_L}$ are given by

$$\begin{aligned}F_{X|\mathcal{A}_L}(x) &= F_{X|\mathcal{A}_L}(x,+\infty)=\frac{F_X(x)}{\alpha},\\ F_{Y|\mathcal{A}_L}(y) &= F_{X|\mathcal{A}_L}(q_\alpha(X),y)=\frac{C(\alpha,F_Y(y))}{\alpha}=\frac{k_\alpha(F_Y(y))}{\alpha},\end{aligned}$$

where $k_\alpha\colon[0,1]\to[0,\alpha]$ denotes the *vertical section* of C at the level α given by $k_\alpha(t)=C(\alpha,t)$. Supposed that $k_\alpha\circ F_Y$ admits inverse, we get

$$C_{\mathcal{A}_L}(u,v)=\frac{C(\alpha u,F_Y(F_Y^{-1}(k_\alpha^{-1}(\alpha v))))}{\alpha}=\frac{C(\alpha u,k_\alpha^{-1}(\alpha v))}{\alpha},$$

for every $(u,v)\in[0,1]^2$. Analogously, we can prove that

$$C_{\mathcal{A}_D}(u,v)=\frac{C(h_\alpha^{-1}(\alpha u),\alpha v)}{\alpha},$$

where $h_\alpha\colon[0,1]\to[0,\alpha]$ denotes the *horizontal section* of C at the level α given by $h_\alpha(t)=C(t,\alpha)$.

Actually, as we have seen from the previous example, the expression of the copula $C_{\mathcal{A}}$ just depends on the value that the copula C of F assumes on some rectangles of the unit square whose vertices are determined by the marginal d.f.'s F_X and F_Y. This fact is formalized in the following result.

Proposition 3. *Let $\mathcal{A}:=[a_1,a_2]\times[b_1,b_2]$ be a rectangle of $\mathbb{R}^2$, with $-\infty\le a_1<a_2\le+\infty$ and $-\infty\le b_1<b_2\le+\infty$. Let $F:\mathbb{R}^2\to\mathbb{R}$ be a*

continuous joint d.f. with marginals F_X and F_Y and copula $C_{(X,Y)}$. Let $\mathcal{S} := [F_X(a_1), F_X(a_2)] \times [F_Y(b_1), F_Y(b_2)]$ be a rectangle of $[0,1]^2$. Then

$$C_{(X,Y)|\mathcal{A}}(u,v) = C_{(U,V)|\mathcal{S}}(u,v) \qquad \text{for all } (u,v) \in [0,1]^2,$$

where $C_{(U,V)|\mathcal{S}}$ is the copula of the conditional d.f. of $[(U,V) \mid (U,V) \in \mathcal{S}]$, C being the joint d.f. of some pair (U,V) of r.v.'s uniformly distributed on $[0,1]$.

Proof. Given two continuous r.v.'s X and Y with joint d.f. F, it follows from the probability integral transformation that $(U,V) := (F_X(X), F_Y(Y))$ is a pair of r.v.'s uniformly distributed on $[0,1]$. For all $(x,y) \in [0,1]^2$, we have

$$\begin{aligned} F_{(U,V)|\mathcal{S}}(x,y) &= \mathbb{P}(U \le x, V \le y \mid (U,V) \in \mathcal{S}) \\ &= \frac{V_C([F_X(a_1), x] \times [F_Y(b_1), y])}{V_C([F_X(a_1), F_X(a_2)] \times [F_Y(b_1), F_Y(b_2)])}. \end{aligned} \tag{8}$$

The marginal d.f.'s of $F_{(U,V)|\mathcal{S}}$ are given, respectively, by

$$\begin{aligned} F_{U|\mathcal{S}}(x) &= \mathbb{P}(U \le x \mid (U,V) \in \mathcal{S}) = \frac{V_C([F_X(a_1), x] \times [F_Y(b_1), F_Y(b_2)])}{V_C([F_X(a_1), F_X(a_2)] \times [F_Y(b_1), F_Y(b_2)])}, \\ F_{V|\mathcal{S}}(y) &= \mathbb{P}(V \le y \mid (U,V) \in \mathcal{S}) = \frac{V_C([F_X(a_1), F_X(a_2)] \times [F_Y(b_1), y])}{V_C([F_X(a_1), F_X(a_2)] \times [F_Y(b_1), F_Y(b_2)])}. \end{aligned}$$

Thus, $F_{U|\mathcal{S}}(F_X(x)) = F_{X|\mathcal{A}}(x)$ and $F_{V|\mathcal{S}}(F_Y(y)) = F_{Y|\mathcal{A}}(y)$. As a consequence, we obtain that

$$F_X(F^{-1}_{X|\mathcal{A}}(u)) = F^{-1}_{U|\mathcal{S}}(u) \quad \text{and} \quad F_Y(F^{-1}_{Y|\mathcal{A}}(v)) = F^{-1}_{V|\mathcal{S}}(v).$$

Now, the copula of $[(U,V) \mid (U,V) \in \mathcal{S}]$ is given by

$$C_{(U,V)|\mathcal{S}}(u,v) = F_{(U,V)|\mathcal{S}}(F_X(F^{-1}_{X|\mathcal{A}}(u)), F_Y(F^{-1}_{Y|\mathcal{A}}(v))),$$

and then, by using (8), we obtain the same expression as in (7). $\square$

Example 2. Let us consider the copula $C_{(X,Y)|\mathcal{A}_{LD}}$, which describes the dependence of a random pair (X,Y) with joint d.f. F when the r.v.'s X and Y fall under the thresholds $q_\alpha(X)$ and $q_\alpha(Y)$, respectively. Thanks to the previous result, this copula is the same as the copula of the random pair (U,V), whose d.f. is the copula C, conditional on the event $\{U \le \alpha, V \le \alpha\}$. Therefore, for every $(x,y) \in [0,\alpha]$, we obtain

$$F_{(U,V)|(U\le\alpha, V\le\alpha)}(x,y) = \frac{C(x,y)}{C(\alpha,\alpha)}.$$

After some computations, we obtain that, for all $(u,v) \in [0,1]$,

$$C_{(X,Y)|\mathcal{A}_{LD}}(u,v) = \frac{C(h_\alpha^{-1}(uh_\alpha(\alpha)), k_\alpha^{-1}(vk_\alpha(\alpha)))}{C(\alpha,\alpha)},$$

where h_α and k_α denote, respectively, the horizontal and vertical section of C at the level α. Detailed investigation on these copulas has been done in [11].

Given a copula C and $\mathcal{A} \subseteq \mathbb{R}^2$, $C_\mathcal{A}$ equals to the copula $C_{(U,V)|\mathcal{S}}$ associated with a pair (U,V) of r.v.'s uniformly distributed on $[0,1]$ with d.f. C conditional to the fact that $(U,V) \in \mathcal{S}$. In particular, the sets $\mathcal{S}'s$ (from the unit square) associated with the previously introduced sets $\mathcal{A}'s$ (from $\mathbb{R}^2$) are given by:

- $\mathcal{S}_L = [0, \alpha] \times [0, 1]$,
- $\mathcal{S}_R = [1 - \alpha, 1] \times [0, 1]$,
- $\mathcal{S}_D = [0, 1] \times [0, \alpha]$,
- $\mathcal{S}_U = [0, 1] \times [1 - \alpha, 1]$,
- $\mathcal{S}_{LD} = [0, \alpha] \times [0, \alpha]$,
- $\mathcal{S}_{RU} = [1 - \alpha, 1] \times [1 - \alpha, 1]$.
- $\mathcal{S}_V = [\beta, 1 - \beta] \times [0, 1]$,
- $\mathcal{S}_H = [0, 1] \times [\beta, 1 - \beta]$,
- $\mathcal{S}_M = [\beta, 1 - \beta] \times [\beta, 1 - \beta]$,

where $\alpha, \beta \in \left]0, \frac{1}{2}\right[$, $\alpha \leq \beta$.

An interesting problem related to threshold copulas from a given set $\mathcal{S}$ is their limit behaviour when $\mathcal{S}$ tends to a degenerate set of 2–Lebesgue measure 0. For threshold copulas $C_{\mathcal{S}_{LD}}$, this limit behaviour has been investigated in [17, 18, 5]. Here, we formulate some related results for the threshold copulas related to central sets.

Proposition 4. *Let C be an absolutely continuous copula with density c. If c is continuous at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $c\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$, then the copula $C_{\mathcal{S}_M}$ converges uniformly to Π when β tends to $\frac{1}{2}$, viz.*

$$\forall (u, v) \in [0, 1]^2 \qquad C_{\mathcal{S}_M}(u, v) \stackrel{\beta \to \frac{1}{2}}{\longrightarrow} uv.$$

Proof. Let (U, V) be a pair of continuous r.v.'s with d.f. C. Consider the set $\mathcal{S}_M = [\beta, 1 - \beta] \times [\beta, 1 - \beta]$. Set $\beta = \frac{1}{2} - t$. Let $F_{\mathcal{S}_M}$ be the conditional distribution function of $[(U, V) \mid \mathcal{S}_M]$. For every $(x, y) \in [-1, 1]^2$ we have

$$(9) \qquad F_{\mathcal{S}_M}\left(\frac{1}{2} + tx, \frac{1}{2} + ty\right) = \frac{V_C([\frac{1}{2} - t, \frac{1}{2} + tx] \times [\frac{1}{2} - t, \frac{1}{2} + ty])}{V_C([\frac{1}{2} - t, \frac{1}{2} + t] \times [\frac{1}{2} - t, \frac{1}{2} + t])} = \frac{\int_{-1}^{y} \int_{-1}^{x} c(\frac{1}{2} + tu, \frac{1}{2} + tv) du dv}{\int_{-1}^{1} \int_{-1}^{1} c(\frac{1}{2} + tu, \frac{1}{2} + tv) du dv}.$$

For $t \to 0$, we obtain that the above expression tends to

$$\frac{\int_{-1}^{y} \int_{-1}^{x} c(\frac{1}{2}, \frac{1}{2}) du dv}{\int_{-1}^{1} \int_{-1}^{1} c(\frac{1}{2}, \frac{1}{2}) du dv} = \frac{(x + 1)(y + 1)c(\frac{1}{2}, \frac{1}{2})}{4c(\frac{1}{2}, \frac{1}{2})} = \frac{(x + 1)(y + 1)}{4}.$$

This fact can be seen by using the Dominated Convergence Theorem and the continuity of c at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$,

$$c\left(\frac{1}{2} + tu, \frac{1}{2} + tv\right) \stackrel{t \to 0}{\longrightarrow} c\left(\frac{1}{2}, \frac{1}{2}\right),$$

for all $(u, v) \in [-1, 1]^2$. Since the conditional distribution function $F_{\mathcal{S}_M}$ tends to the product of univariate distribution functions (after a linear transformation of its arguments as in (9)), the limiting copula of $C_{\mathcal{S}_M}$ is Π. $\square$

Proposition 5. *Let C be an absolutely continuous copula with density c. If c is continuous at all points of the set $\left\{\frac{1}{2}\right\} \times [0,1]$, then the copula $C_{\mathcal{S}_V}$ converges uniformly to Π when $\beta \to \frac{1}{2}$, viz.*

$$\forall (u,v) \in [0,1]^2 \qquad C_{\mathcal{S}_V}(u,v) \stackrel{\beta \to \frac{1}{2}}{\longrightarrow} uv.$$

Analogous result can be formulated for $C_{\mathcal{S}_H}$.

Proof. As in the previous proof, let (U,V) be a pair of continuous r.v.'s with d.f. C. Consider the set $\mathcal{S}_V = [\beta, 1-\beta] \times [0,1]$. Set $\beta = \frac{1}{2} - t$. Let $F_{\mathcal{S}_V}$ be the conditional distribution function of $[(U,V) \mid \mathcal{S}_V]$. For every $(x,y) \in [-1,1] \times [0,1]$, we have

$$F_{\mathcal{S}_V}\left(\frac{1}{2} + tx, y\right) = \frac{V_C([\frac{1}{2} - t, \frac{1}{2} + tx] \times [0,y])}{2t} = \frac{1}{2} \int_0^y \int_{-1}^x c\left(\frac{1}{2} + tu, v\right) du dv. \tag{10}$$

As $t \to 0$, the above expression tends to

$$\frac{1}{2} \int_0^y \int_{-1}^x c\left(\frac{1}{2}, v\right) du dv = \frac{x+1}{2} \int_0^y c\left(\frac{1}{2}, v\right) dv,$$

for all $(u,v) \in [-1,1] \times [0,1]$. This fact can be seen by using the Dominated Convergence Theorem and the continuity of c at all points of the set $\left\{\frac{1}{2}\right\} \times [0,1]$. Since the conditional distribution function $F_{\mathcal{S}_V}$ tends to the product of univariate distribution functions (after a linear transformation of its first argument as in (10)), then the limiting copula of $C_{\mathcal{S}_V}$ is Π. □

3. Definitions of contagion

As we have said in the introduction, the notion of (spatial) contagion is related to the comparison of the dependence among two financial markets X and Y in some specific regions of their joint distributions. At a more theoretical level, all these concepts can be translated in terms of comparisons among d.f.'s and conditional d.f.'s with respect to some meaningful tail and central events, or, equivalently, in terms of copulas and threshold copulas.

The most common way to compare the strenghtness of dependence among to random pairs is to consider the *positive quadrant dependence* (shortly, PQD) ordering between their respective copulas. We recall that, given two copulas C_1 and C_2, we say that $C_1 \preceq_{PQD} C_2$ if, for all $(u,v) \in [0,1]^2$, $C_1(u,v) \leq C_2(u,v)$. In particular, we adopt the symbol $\prec_{PQD}$ in order to indicate the case when $C_1 \preceq_{PQD} C_2$ but $C_1 \neq C_2$ [16, 9]. With respect to this ordering, we have that, for any copula C, $W \preceq C \preceq M$. Moreover, a copula C is said to be *positive quadrant dependent* if $C \succeq \Pi$.

Now, we are able to formulate the following definitions of contagion. Let X and Y be the r.v.'s representing returns of two financial markets. Suppose that C is the copula of (X,Y).

Definition 1. We say that there is *contagion* from X to Y with respect to $\mathcal{A}_L$ and $\mathcal{A}_V$ if

$$C_{\mathcal{A}_V} \prec_{PQD} C_{\mathcal{A}_L}.$$

Analogously, we say that there is *contagion* from Y to X with respect to $\mathcal{A}_D$ and $\mathcal{A}_H$ if

$$C_{\mathcal{A}_H} \prec_{PQD} C_{\mathcal{A}_D}.$$

We say that there is *symmetric contagion* between X and Y with respect to $\mathcal{A}_{LD}$ and $\mathcal{A}_M$ if

$$C_{\mathcal{A}_M} \prec_{PQD} C_{\mathcal{A}_{LD}}.$$

Thus, contagion is defined as an increase of the dependence in some tail regions of the joint distribution of (X, Y) with respect to some central regions. Moreover, as just copulas describe the dependence among r.v.'s, contagion refers to the comparison among threshold copulas obtained with respect to tail regions or central regions of the unit square.

Note that contagion appears when there is a strict order relation among the two copulas. In other words, contagion appears when the dependence increases *significantly* in a given tail region with respect to the center of the distribution (compare with [15]). For instance, two comonotone markets exhibits no contagion. In fact, at any state of the worlds, their copula is always the comonotone copula $M(u, v) = \min(u, v)$ and the dependence does not increase. Note that different kinds of orderings among copulas may be used in Definition 1 (like LTD, TP2, SI) [16, 9].

This kind of comparison is more informative than other comparison concerning special indices associated to joint d.f.'s like tail dependence coefficients [13, 23, 20]. In fact, it is well recognized that distributions with same tail dependence coefficients could have a quite different behaviour in their tails.

Remark 1. The given definitions could be slightly modified to consider the changes in dependence not when the markets are doing badly, but when they have good performances. So, in a situation that we can call *berserk mode* (investors become fearless). However, we would like in the present paper just to restrict to the classical case: all methodologies can be straightforward applied also to the other situation, as we will see on section 4.

In the following examples, we illustrate, by means of copulas, several theoretical "exotic" case of dependence that financial markets may have. All these examples are constructed by using rectangular patchwork techniques [12]. We recall that, given a copula C and a finite family $\{R_i\}$ of rectangles of $[0, 1]^2$ such that, whenever $i \neq j$, R_i and R_j have common points just on their boundaries, a *rectangular patchwork* is any copula $F : [0, 1]^2 \to [0, 1]$ defined by

$$F(u, v) = \begin{cases} F_i(u, v), & (u, v) \in R_i, \\ C(u, v), & \text{otherwise}, \end{cases} \tag{11}$$

for suitable function $F_i \colon R_i \to [0, 1]$. Essentially, a rectangular patchwork is obtained by modifying the values that a copula C assumes on some subdomains of $[0, 1]^2$ in such a way that the resulting function is again a copula. A special example of rectangular patchwork is the ordinal sum construction [21].

Example 3. Let $\alpha \in \left]0, \frac{1}{2}\right[$ and let C be the copula given by:

$$C(u,v) = \begin{cases} \alpha^2 M\left(\frac{u}{\alpha}, \frac{v}{\alpha}\right), & (u,v) \in [0,\alpha]^2, \\ uv, & \text{otherwise,} \end{cases}$$

where $\alpha \in \left]0, \frac{1}{2}\right[$, $\overline{\alpha} = 1 - \alpha$, and M is the comonotone copula. Then r.v.'s X and Y with copulas C exhibits symmetric contagion. In fact, after some little algebra, we obtain $C_{\mathcal{S}_M} = \Pi$ for every $\beta \geq \alpha$, but $C_{\mathcal{S}_{LD}} = M$.

Example 4. Let $\alpha \in \left]0, \frac{1}{2}\right[$ and let us consider the copula D given by

$$D(u,v) = \begin{cases} \alpha C\left(\frac{u}{\alpha}, \frac{v}{\alpha}\right), & (u,v) \in [0,\alpha]^2, \\ \min(u,v), & \text{otherwise,} \end{cases}$$

where $\alpha \in \left]0, \frac{1}{2}\right[$ and C is another copula, $C \neq M$ then there is no symmetric contagion among r.v.'s X and Y with copula D. In fact, for every $\beta \geq \alpha$, $D_{\mathcal{S}_M} = M$, but $D_{\mathcal{S}_{LD}} = C \prec M$. Note the copula C can be chosen such that D has a strong dependence in its lower left tail. However, as we have said, this does not imply automatically that there is some kind of contagion.

Example 5. There are random variables X and Y such that there is contagion from X to Y, but not viceversa. Suppose, for example, that X and Y are connected by the copula

$$C(u,v) = \begin{cases} \min(u - \alpha u, \alpha v - \alpha^2) + \alpha u, & (u,v) \in [0,\alpha] \times [\alpha, 1], \\ uv, & \text{otherwise,} \end{cases}$$

for some $\alpha \in \left]0, \frac{1}{2}\right[$. Then, we have that

$$C_{\mathcal{S}_L}(u,v) = \begin{cases} uv, & v \leq \alpha, \\ \min(u(1-\alpha), v - \alpha) + \alpha u, & v > \alpha. \end{cases}$$

Thus $C_{\mathcal{S}_L} \succ C_{\mathcal{S}_V} = \Pi$ for every $\alpha \leq \beta$, but $C_{\mathcal{S}_D} = \Pi = C_{\mathcal{S}_H}$. Situation of asymmetric contagion, i.e. contagion from one market to the other, typically are related to the case when a financial market entering a bearish period may affect others markets, even if they have apparently stronger fundamentals economic conditions.

Note that, if C is symmetric, then contagion from X to Y implies contagion from Y to X. However, this is not equivalent to the fact that there is a symmetric contagion between X and Y, even if the copula is symmetric.

Example 6. Let $\alpha \in \left]0, \frac{1}{2}\right[$ and let us consider the copula C given by

$$C(u,v) = \begin{cases} \alpha^2 \min\left(\dfrac{u - \overline{\alpha}}{\alpha}, \dfrac{v}{\alpha}\right) + \overline{\alpha} v, & (u,v) \in [\overline{\alpha}, 1] \times [0, \alpha], \\ \alpha^2 \min\left(\dfrac{u}{\alpha}, \dfrac{v - \overline{\alpha}}{\alpha}\right) + \overline{\alpha} u, & (u,v) \in [0,\alpha] \times [\overline{\alpha}, 1], \\ uv, & \text{otherwise,} \end{cases}$$

where $\overline{\alpha} = 1 - \alpha$. Then, for every $\beta \geq \alpha$, we have $C_{\mathcal{S}_M} = \Pi = C_{\mathcal{S}_{LD}}$. However,

$$C_{\mathcal{S}_L} = \begin{cases} uv, & v \leq \overline{\alpha}, \\ \min(\alpha u, v - \overline{\alpha}) + \overline{\alpha} u, & v > \overline{\alpha}, \end{cases}$$

and

$$C_{\mathcal{S}_D} = \begin{cases} uv, & u \leq \overline{\alpha}, \\ \min(u - \overline{\alpha}, v) + \overline{\alpha} v, & u > \overline{\alpha}. \end{cases}$$

Thus, little algebra shows that $\Pi \prec C_{\mathcal{S}_L}$ and $\Pi \prec C_{\mathcal{S}_D}$.

Our definitions of contagion (and that one in [2]) do not require any explicit definition of crisis and normal periods. However, these notions appear, in some sense, in the background. In fact, the choice of the appropriate tail set, and hence of the appropriate $\alpha \in \left]0, \frac{1}{2}\right[$, implies *de facto* that we suppose that there is a crisis period when a r.v. falls under its α–quantile. Analogously, the choice of the central set, and hence of the appropriate $\beta \in \left]0, \frac{1}{2}\right[$, determines which values assumed by the r.v.'s can be considered as "normal" (i.e. just "standard" fluctuations in the market).

While the definition of some given threshold for the tail of the distribution is current practice in finance (just think at the value at risk), the definition of the thresholds for the central set seems to be more arbitrary. In order to avoid this, we could also compare the threshold copula of a tail set with the limit threshold copula of a central set when $\beta \to \frac{1}{2}$ (if this limit exists). In these cases, we will speak more properly of *asymptotic contagion* at a given level α (related to the tail set).

Definition 2. We say that there is *asymptotic contagion* from X to Y at the level α if

$$\lim_{\beta \to 1/2} C_{\mathcal{A}_V} \prec_{PQD} C_{\mathcal{A}_L}.$$

Analogously, we say that there is *asymptotic contagion* from Y to X at the level α if

$$\lim_{\beta \to 1/2} C_{\mathcal{A}_H} \prec_{PQD} C_{\mathcal{A}_D}.$$

We say that there is *asymptotic symmetric contagion* between X and Y at the level α if

$$\lim_{\beta \to 1/2} C_{\mathcal{A}_M} \prec_{PQD} C_{\mathcal{A}_{LD}}.$$

Under some assumptions on the copula of (X, Y), we can obtain the following results, whose proofs just depend on Propositions 4 and 5.

Corollary 1. *Under the assumptions of Proposition 5, there is asymptotic contagion from X to Y at the level $\alpha \in \left]0, \frac{1}{2}\right[$ if $\Pi \prec_{PQD} C_{\mathcal{A}_L}$.*
Analogous results can be formulated for asymptotic contagion from Y to X.

Corollary 2. *Under the assumptions of Proposition 4, there is asymptotic symmetric contagion between X and Y at the level $\alpha \in \left]0, \frac{1}{2}\right[$ if $\Pi \prec_{PQD} C_{\mathcal{A}_{LD}}$.*

Therefore, in these cases, asymptotic contagion with respect to the tail set $\mathcal{A}$ just depends from the fact that $C_{\mathcal{A}} \succ \Pi$, which is just a strict version of the positive quadrant dependence property [21].

Proposition 6. *Given assumptions of Proposition 5 and 4, respectively, and given $\alpha \in \left]0, \frac{1}{2}\right[$, the following statements hold.*

(i) *If, for all $(u,v) \in [0,\alpha] \times [0,1]$,*

$$\alpha C(u,v) \le u C(\alpha, v), \tag{12}$$

and the above inequality is strict for at least one $(u_0, v_0) \in [0,\alpha] \times [0,1]$, then $\Pi \prec_{PQD} C_{\mathcal{S}_L}$.

(ii) *If, for all $(u,v) \in [0,\alpha]^2$,*

$$C(u,v)C(\alpha,\alpha) \ge C(u,\alpha)C(\alpha,v), \tag{13}$$

and the above inequality is strict for at least one $(u_0, v_0) \in [0,\alpha]^2$, then $\Pi \prec_{PQD} C_{\mathcal{S}_{LD}}$.

Proof. By definition, $\Pi \prec_{PQD} C_{\mathcal{S}_L}$ if, and only if, $C \neq \Pi$ and for all $(u,v) \in [0,1]^2$,

$$uv \le C_{\mathcal{S}_L}(u,v) = \frac{C(\alpha u, k_\alpha^{-1}(\alpha v))}{\alpha},$$

where k_α denotes the *vertical section* of C at the level α. Thus, for every $(u,v) \in [0,\alpha] \times [0,1]$, we get

$$\alpha C(s,t) < s k_\alpha(s) = s C(\alpha, t).$$

Part (ii) follows directly from [11, Proposition 9]. □

Note that inequality (12) (respectively, inequality (13)) implies that, for every $(u,v) \in \mathcal{S}_L$ (respectively $\mathcal{S}_{LD}$), $C(u,v) \ge uv$, being the converse implication not generally true [11]. Moreover, if C is TP2 on $\mathcal{S}_L$, viz. $C(u_1,v_1)C(u_2,v_2) \ge C(u_1,v_2)C(u_2,v_1)$ for all $0 \le u_1 \le u_2 \le 1$ and $0 \le v_1 \le v_2 \le 1$, then C satisfies (12). Same results can be formulated for the other case.

Example 7. Let $\alpha \in \left]0, \frac{1}{2}\right[$ and let C be the copula given by

$$C(u,v) = \begin{cases} \alpha C_1\left(\frac{u}{\alpha}, \frac{v}{\alpha}\right), & (u,v) \in [0,\alpha]^2, \\ \alpha + (1-2\alpha) C_2\left(\frac{u-\alpha}{1-2\alpha}, \frac{v-\alpha}{1-2\alpha}\right), & (u,v) \in [\alpha, 1-\alpha]^2, \\ \min(u,v), & \text{otherwise}. \end{cases}$$

where C_1 and C_2 are two members of the FGM copulas given respectively by $C_1(u,v) = uv\left(1 + \frac{1}{2}(1-u)(1-v)\right)$ and $C_2(u,v) = uv\left(1 + (1-u)(1-v)\right)$. Then, $C_{\mathcal{S}_{LD}}$ satisfies (13) (because C_1 is TP2 on $[0,1]^2$) and, because we are in the conditions of applying Proposition 4, there is asymptotic contagion at level α. However, there is no symmetric contagion with respect to $\mathcal{S}_{LD}$ and $\mathcal{S}_M = [\alpha, 1-\alpha] \times [\alpha, 1-\alpha]$. In fact $C_{\mathcal{S}_M} = C_2 \succ C_{\mathcal{S}_{LD}} = C_1$.

4. An empirical illustration

Let X and Y be the r.v.'s representing returns related to two financial markets. Suppose that we have at disposal some data corresponding to historical time series from X and Y, say $(x_i)_{i=1}^N$ and $(y_i)_{i=1}^N$. In order to verify whether, on the basis of such data, there is some kind of contagion among X and Y, two ways can be applied.

The first one consists in estimating the *copula* $\widehat{C}$ of (X, Y) and, hence, trying to get the explicit formulas for the related threshold copulas by using the results of section 2. Although theoretically correct, this procedure is of problematic implementation, due to at (least) two reasons. Firstly, the problem of fitting a "good" copulas to some real empirical data always suffer of some error estimations that may be also amplified in the computation of the related threshold copulas. Secondly, the calculation of the threshold copulas is sometimes a difficult task and no closer forms can be obtained.

Here, we propose a different non–parametric procedure. As we have seen, all the definitions of contagion are based on the comparisons among copulas in the PQD ordering. This ordering is also known to be a concordance ordering in the sense that, if $C \preceq_{PQD} D$, then $\kappa(C) \leq \kappa(D)$, where κ is any measure of concordance (for instance, Kendall's τ or Spearman's ρ) [25, 21]. Therefore, in practice, we will check for contagion in the following week sense.

Definition 3. Let κ be a measure of concordance. Let X and Y be two financial markets connected by the copula C. We say that:

- there is *contagion* from X to Y with respect to $\mathcal{S}_L$ and $\mathcal{S}_V$ if
$$\kappa(C_{\mathcal{S}_V}) < \kappa(C_{\mathcal{S}_L});$$
- there is *contagion* from Y to X with respect to $\mathcal{S}_D$ and $\mathcal{S}_H$ if
$$\kappa(C_{\mathcal{S}_H}) < \kappa(C_{\mathcal{S}_D});$$
- there is *symmetric contagion* between X and Y with respect to $\mathcal{S}_{LD}$ and $\mathcal{S}_M$ if
$$\kappa(C_{\mathcal{S}_M}) < \kappa(C_{\mathcal{S}_{LD}}).$$

In the following, as measure of concordance we will consider the Spearman's rank correlation ρ. In particular, the Sperman's ρ of some threshold copula $C_{\mathcal{S}}$ is usually known as *conditional* Spearman's ρ and it has been investigated in [8].

To illustrate the practical usefulness of our approach we shall consider two markets: the New York Stock Exchange (US) and SWX Swiss Exchange AG in Zurich (Switzerland). We shall compare the daily log-returns of the indices – Dow Jones Industrial Average (DJIA) and Swiss Market Index (SMI) related to the period 9th November 1990 – 14th April 2008. Of course, we consider only the days when both market were operating (4281 observations). By using standard procedure, we calculate the Spearman's ρ coefficient for the empirical threshold copulas. Then, we estimate the confidence intervals at the level 0.8 (i.e., we cut 0.1 of each tail). This interval is obtained by resampling from the copula $\widehat{C}$, obtained by bilinear extension of the empirical copula to the whole $[0, 1]^2$ [21]. Several cases have been considered for testing whether there is contagion, by considering the one–side test like

$$\begin{array}{lll} H_0: & \rho(C_{\mathcal{T}}) > \rho(C_{\mathcal{M}}) & \text{(contagion)} \\ H_1: & \rho(C_{\mathcal{T}}) \leq \rho(C_{\mathcal{M}}) & \text{(no contagion)} \end{array}$$

for some suitable tail set $\mathcal{T}$ and central set $\mathcal{M}$.

• Symmetric contagion.
We consider the threshold copulas for the following sets:

- $\mathcal{S}_M(0.05) = [q_{0.05}(X), q_{0.95}(X)] \times [q_{0.05}(Y), q_{0.95}(Y)]$,
- $\mathcal{S}_M(0.30) = [q_{0.3}(X), q_{0.7}(X)] \times [q_{0.3}(Y), q_{0.7}(Y)]$,
- $\mathcal{S}_{LD}(0.05) = \,]-\infty, q_{0.05}(X)] \times \,]-\infty, q_{0.05}(Y)]$,
- $\mathcal{S}_{RU}(0.05) = \,[q_{0.95}(X), \infty[\times [q_{0.95}(Y), \infty[$,

Set	No. of samples	$\rho(C_{\mathcal{S}})$	Confidence Interval
$\mathcal{S}_M(0.05)$	3565	0.207	$[0.185, 0.228]$
$\mathcal{S}_M(0.30)$	795	0.030	$[-0.016, 0.078]$
$\mathcal{S}_{LD}(0.05)$	65	0.341	$[0.178, 0.485]$
$\mathcal{S}_{RU}(0.05)$	64	0.428	$[0.308, 0.577]$

Table 1. Symmetric contagion

As we can see in Table 1,

$$\rho(C_{\mathcal{S}_M(0.30)}) < \rho(C_{\mathcal{S}_{LD}(0.05)}) \text{ and } \rho(C_{\mathcal{S}_M(0.30)}) < \rho(C_{\mathcal{S}_{RU}(0.05)}),$$
$$\rho(C_{\mathcal{S}_M(0.05)}) < \rho(C_{\mathcal{S}_{LD}(0.05)}) \text{ and } \rho(C_{\mathcal{S}_M(0.05)}) < \rho(C_{\mathcal{S}_{RU}(0.05)}).$$

By analyzing the related confidence intervals, we cannot reject the assumption of symmetric contagion (respectively, symmetric berserk mode) with respect to $\mathcal{S}_{LD}(0.05)$ (respectively, $\mathcal{S}_{RU}(0.05)$) and $\mathcal{S}_M(0.30)$. On the other hand, we have to reject the assumption of symmetric contagion with respect to $\mathcal{S}_{LD}(0.05)$ and $\mathcal{S}_M(0.30)$, because the difference among the corresponding Spearman's ρ is not statistically significant. But, we cannot reject the assumption of symmetric berserk mode with respect to $\mathcal{S}_{RU}(0.05)$ and $\mathcal{S}_M(0.05)$.

• Contagion from DJIA to SMI.
We consider the threshold copulas for the following sets:

- $\mathcal{S}_V(0.05) = [q_{0.05}(X), q_{0.95}(X)] \times \mathbb{R}$,
- $\mathcal{S}_V(0.30) = [q_{0.3}(X), q_{0.7}(X)] \times \mathbb{R}$,
- $\mathcal{S}_L(0.05) = \,]-\infty, q_{0.05}(X)] \times \mathbb{R}$,
- $\mathcal{S}_R(0.05) = \,]q_{0.95}(X), \infty] \times \mathbb{R}$,

Set	No. of samples	$\rho(C_{\mathcal{S}})$	Confidence Interval
$\mathcal{S}_V(0.05)$	3852	0.248	$[0.228, 0.268]$
$\mathcal{S}_V(0.30)$	1712	0.050	$[0.019, 0.081]$
$\mathcal{S}_L(0.05)$	214	0.220	$[0.128, 0.309]$
$\mathcal{S}_R(0.05)$	215	0.332	$[0.242, 0.417]$

Table 2. Contagion from DJIA to SMI

As in Table 2

$$\rho(C_{\mathcal{S}_V(0.30)}) < \rho(C_{\mathcal{S}_L(0.05)}) \text{ and } \rho(C_{\mathcal{S}_V(0.30)}) < \rho(C_{\mathcal{S}_R(0.05)}).$$

By analyzing the related confidence intervals, we cannot reject the assumption of contagion from DJIA to SMI (respectively, berserk mode) with respect to $\mathcal{S}_{LD}(0.05)$ (respectively, $\mathcal{S}_{RU}(0.05)$) and $\mathcal{S}_M(0.30)$.

On the other hand, $\rho(C_{\mathcal{S}_V(0.05)}) > \rho(C_{\mathcal{S}_L(0.05)})$ implies that we have to reject the assumption of contagion with respect to $\mathcal{S}_L(0.05)$ and $\mathcal{S}_V(0.05)$. Moreover, we have to reject the assumption of berserk mode with respect to $\mathcal{S}_R(0.05)$ and $\mathcal{S}_V(0.05)$, because the difference among the corresponding Spearman's ρ is not statistically significant.

• Contagion from SMI to DJIA.
We consider the threshold copulas for the following sets:

- $\mathcal{S}_H(0.05) = \mathbb{R} \times [q_{0.05}(Y), q_{0.95}(Y)]$,
- $\mathcal{S}_H(0.30) = \mathbb{R} \times [q_{0.3}(Y), q_{0.7}(Y)]$,
- $\mathcal{S}_D(0.05) = \mathbb{R} \times]-\infty, q_{0.05}(X)]$,
- $\mathcal{S}_U(0.95) = \mathbb{R} \times]q_{0.95}(X), \infty]$,

Set	No. of samples	$\rho(C_\mathcal{S})$	Confidence Interval
$\mathcal{S}_H(0.05)$	3852	0.248	$[0.229, 0.270]$
$\mathcal{S}_H(0.30)$	1712	0.077	$[0.042, 0.113]$
$\mathcal{S}_D(0.05)$	214	0.232	$[0.125, 0.307]$
$\mathcal{S}_U(0.05)$	215	0,142	$[0.048, 0.232]$

TABLE 3. Contagion from SMI to DJIA

As in Table 3

$$\rho(C_{\mathcal{S}_H(0.30)}) < \rho(C_{\mathcal{S}_D(0.05)}) \text{ and } \rho(C_{\mathcal{S}_H(0.30)}) < \rho(C_{\mathcal{S}_U(0.05)}).$$

By analyzing the related confidence intervals, we cannot reject the assumption of contagion with respect to $\mathcal{S}_D(0.05)$ and $\mathcal{S}_M(0.30)$; but we have to reject the berserk mode with respect to $\mathcal{S}_U(0.05)$ and $\mathcal{S}_M(0.30)$, because the difference among the corresponding Spearman's ρ is not statistically significant.

On the other hand, $\rho(C_{\mathcal{S}_H(0.05)}) > \rho(C_{\mathcal{S}_D(0.05)})$ and $\rho(C_{\mathcal{S}_H(0.05)}) > \rho(C_{\mathcal{S}_U(0.05)})$ imply that we have to reject the assumption of contagion and berserk mode at these sets.

5. Conclusions

We have discussed some definitions of contagion among two financial markets X and Y. Our definitions are completely based on the knowledge of the copula C among X and Y, and depend on the values that C assumes on some specific regions. For these purpose, the concept of threshold copulas are provided to be useful. These definitions allow a practical investigation whether there is contagion among X and Y both in a parametric and in a non–parametric setting.

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