Bid-ask spread modelling: a perturbation approach *

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Abstract

Our objective is to study liquidity risk, in particular the so-called "limit order books", as a by-product of market uncertainties. "Limit order books" describe the existence of different sell and buy prices, which we explain by using different risk aversions of the agents. The risky assets follows a local volatility diffusion governed by a Brownian motion which is uncertain. We use the error theory with Dirichlet forms to formalise the notion of uncertainty on the Brownian motion. This uncertainty generates a noise on the trajectories of the underlying assets and we use this noise to expound the presence of a bid-ask spread. In addition, we prove that this noise also has a direct impact on mid-price of risky assets. We enrich our analysis with a numerical simulation when the volatility is a power function of the asset price.

Extended abstract

Classical market models in mathematical finance assume perfect elasticity of traded assets : traders act as price takers, so that they buy and sell with arbitrary size without changing the price. They further assume that traders may buy or sell stocks at the same prices, in other words, there is no spread between the bid and ask prices, which are both equal. However, as shown in the market microstructure literature, it is clear that large trades move the price of the assets. Moreover, the spread between the bid and ask prices does exist and is intrinsic to the financial markets.

There are several approaches in modelling liquidity risk. One approach is to explain liquidity risk by the presence of an insider [3], [12]. Another approach is the market manipulation literature where prices are assumed to depend directly on the trading strategies, see [10], [6] and [8]. A third approach is to consider liquidity risk in terms of the difference between the bid and ask prices, i.e. the existence of a bid-ask spread. Transaction costs [11] are one way to model the bid-ask spread. In [13], liquidity risk is expressed by the presence of transaction costs and market manipulation. Another way is to directly model the bid-ask spread or the order books and explain its presence as being intrinsic to the financial markets, driven by trades between different market participants.

Bid-ask spread and order books play a crucial role in many financial problems such as unwinding large block order of shares for large investors and hedging strategy of options for traders. These problems were investigated by the likes of Bertsimas and Loo [4], Almgren et

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al. [2] Obizhaeva and Wang [14], Alfonsi et al. [1]. Due to the complexity of the studies, state process representing the underlying price, is assumed to have simple dynamics such as Bachelier's dynamics [14], or is assumed to be a martingale process [1].

However, in the above studied models, liquidity risk is considered a posteriori. In other words, assuming the existence of liquidity risk, different approaches are used to replicate its effects. To our knowledge, few studies in the fields of mathematical finance have attempted to model the financial and economic rationales behind the existence of the bid-ask spread. This is precisely the objective in this paper: study and explain the existence of the bid-ask spread, and more generally limit order book, as a by-product of market uncertainties.

It is well-known that the price of an assets is theoretically the discounted of expected future cash flows, which are random processes and must be estimated. Thus, the value of an assets is an estimation obtained under uncertainty and may therefore be represented by a random variable. In other words, at any fixed time, the value of an assets is not observable but a random variable where its law or at least its mean value and variance may be characterized.

In conformity with this point of view, the presence of many sell and buy order prices can be explained by different risk aversions of market participants. In order to clarify this idea, we consider a "representative" market maker in a quote-driven market, who has to place both a buy and a sell limit orders, i.e. the prices and the number of shares he is willing to buy and sell. Prior to setting the buy and sell orders, the market maker obtains the distribution of possible asset values from the market information but has no possibility to observe the assets' realized values. A rational decision is to send a limit buy (sell) order with a price lower (higher) with respect to the asset mean value such that their difference justifies the risk taken. Of course, he adjusts those prices by increasing them if he runs short of stock, or cutting them if he starts accumulating excessive stocks.

The mathematical formulation of such problems relies on the specification of a coherent framework to describe the remaining randomness on prices. As a matter of fact, in our problem, the asset value must depend on two random sources: the first one describes the evolution of the asset mean value while the second delineates the shape of asset (sell-buy) prices at a given fixed time. The coupling of the two probability spaces, with its respective filtration, requires complex tools and represents the principal drawback of this kind of approach. Therefore, we choose a different strategy based on error theory using Dirichlet forms formalism. The advantages of this approach are inherent to its elasticity and powerful tools. Order books framework justifies automatically many assumptions of error theory, e.g. bid-ask spreads are almost always very negligible with respect to the mid price, allowing the limit expansion approach.

An important peculiarity of order books, and equally a drawback, is the lack of information. Indeed, order books are not completely public, e.g, in France, the market regulator restricts the publicly known part of a book to the five best prices. As such, it is impossible to define the shape of prices randomness with only ten data[1]. In our analysis, we assume that the shape of the book has a gaussian behavior. This is another assumption related to error theory which seems, however, to agree with empirical observations made by Biais et al. [5] and Potters et al. [15], who show a decreasing shape of order books as price goes away from the mid-price. Yet, the maximum is reached near but not always precisely on the first best ask (bid) price. This shifting on the maximum is hard to justify in a gaussian framework, since the maximum for gaussian density is reached at the mean. Error theory provides us with perfect tools to deal with this problem as it foresees a bias with respect to the theoretical mean. This discrepancy can explain the shifting on the maximum.

[1] The five best sell/buy prices

The model

We aim at modelling the dynamics of the mid-price and the bid and ask prices. We consider the following local volatility model:

$$dX_t = X_t\, r_t\, dt + \sigma(t,\, X_t)\, dW_t \tag{0.1}$$

The goal, in financial applications, is to take into account the presence of a difference between the ask and the bid price of an assets. We assume that X_t represents asset mid-price, while the ask and the bid prices are a consequence of the presence of an uncertainty on Brownian motion W_t. We model this uncertainty thanks to error theory using Dirichlet forms, see Bouleau [7]. The mathematical formulation of this uncertainty is done through a second independent Brownian motion with a very small wideness that perturbs the first Brownian motion W_t, resulting in a noise around the mid price X_t.

Result 0.1 (Bid-Ask Spread) *The uncertainty on Brownian motion is transmitted to the stochastic process X_t, that represents the asset price. Therefore, each realization $\overline{\omega}$ of process X_t at time t is not a fixed value $X_t(\overline{\omega})$ but it is a random variable described by*

$$X_t(\overline{\omega}) + \epsilon\, A[X_t(\overline{\omega})] + \sqrt{\epsilon\, \Gamma[X_t(\overline{\omega})]}\, \widetilde{\mathcal{N}}(0,\, 1) \tag{0.2}$$

where

- *ϵ is a small parameter;*
- *$\Gamma[X_t]$ and $A[X_t]$ are two univocal stochastic processes, depending on X_t and characterized by an explicit closed-form;*
- *$\widetilde{\mathcal{N}}(0,\, 1)$ is an independent reduced gaussian random variable.*

Now we consider the presence of many agents in the market, who are informed about the economic evolution of the mid-price X_t but without money-market intelligence about the residual information drawn by the perturbation. All agents are risk adverse and can estimate the distribution of the uncertainty of asset price at each fixed time t. It stands to reason that, at each time t, there exists an agent with minimal risk aversion with respect to his colleagues. This agent accepts to buy the assets at a price B_t smaller that X_t, owing to risk aversion, but the biggest compared to the bid prices of his colleagues. Thus, B_t is the bid price. A symmetric analysis generates the ask price A_t.

Let us assume that there exists a representative agent that proposes always the best buy and sell prices and we assume that this agent accepts to buy the assets at a price B_t such that the risk of overvaluing of the assets is equal to a supportable risk probability $\chi < 0.5$, clearly the "risk" of undervaluing is $1 - \chi > 0.5$, therefore the agent take the risk against the expected earnings. The definition of ask price A_t is symmetric. Finally, error theory foresees a bias that we have to evaluate, this bias shift the mid-price with respect to the theoretical price X_t.
We obtain the second following result:

Result 0.2 *Risk aversion theory permits to define a supportable risk probability $\chi < 0.5$ such that an agent accepts to buy the stock at price*

$$B_t(\overline{\omega}) = (Bid\ price)(t,\, \overline{\omega}) = X_t(\overline{\omega}) + \epsilon\, A[X_t(\overline{\omega})] + \sqrt{\epsilon\, \Gamma[X_t(\overline{\omega})]}\, \widetilde{\mathcal{N}}(\chi)$$

and, by symmetry, a supportable risk probability $\overline{\chi} = 1 - \chi$ such that an agent accepts to sell the stock at price

$$A_t(\overline{\omega}) = (Ask\ price)(t,\, \overline{\omega}) = X_t(\overline{\omega}) + \epsilon\, A[X_t(\overline{\omega})] + \sqrt{\epsilon\, \Gamma[X_t(\overline{\omega})]}\, \widetilde{\mathcal{N}}(1 - \chi)$$

Liquidity model

Up to now, the liquidity of the assets is frozen in our model, since the risk aversion of the trader is constant and does not depend on any parameter such as its trading volume or other market liquidity estimation.

In order to extend our model to take into account liquidity evolution, we propose to differentiate χ_a and χ_b, i.e. ask and bid risk aversions. Furthermore, the two risk aversions become a function of the trading volume, asset prices and other market liquidity information. For instance, tts dependence on the trading volume allows the representative agent or market maker to adjust his bid and ask prices by increasing them if he runs short of stock, or cutting them if he starts accumulating excessive stocks. We suggest to describe the two risk aversions using two stochastic processes in order to evaluate large trader impacts and liquidity crises.

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