An application to credit risk of optimal quantization methods for nonlinear filtering *

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Abstract

The main contribution of this work is to use optimal quantization methods in order to approximate the conditional survival probabilities of a firm.
We consider the case where the value of the firm is a non-observable stochastic process $(V_t)_{t\geq 0}$ and investors in the market have access to the quotations of an asset $(S_t)_{0\leq t\leq T}$ issued by the firm.
We are interested in the computation of the conditional survival probability of the firm from an investor's point of view, i.e., given the "investor information".
We show that solving this problem, under partial information, leads to a nonlinear-filtering problem and our contribution is the use of optimal quantization methods, as proposed in [3], to obtain the discrete time approximation of the filter distribution and the desired estimations for the survival probabilities.
We compare our results with the ones presented in [1] and [2] and we analyze the shape of the credit spread curve in some examples.

Executive summary

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space representing all the randomness of our economic context. Suppose that the value of the firm process $(V_t)_{t\geq 0}$, given for example by its value of financial statement, is the solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}V_t &= b(t,V_t)\mathrm{d}t + \sigma(t,V_t)\mathrm{d}W_t, \\ V_0 &= v_0, \end{cases} \tag{0.1}$$

where $b : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ are Lipschitz in x, uniformly in t, and W is a standard one-dimensional Brownian motion.
In our setting V (the *state* or *signal*) is non observable, but investors have access to the values of another stochastic process S, which can be thought as the price of an asset issued by the firm and that follows a diffusion of the type

$$\begin{cases} \mathrm{d}S_t &= S_t \left[\mu_t V_t \mathrm{d}t + \nu_t \rho \mathrm{d}W_t + \nu_t \sqrt{1-\rho^2}\mathrm{d}\bar{W}_t\right], \\ S_0 &= s_0, \end{cases} \tag{0.2}$$

where $\bar{W}$ is a one-dimensional Brownian motion independent of W (note that the logarithm of S depends linearly on V, meaning that the return of the asset is directly proportional to the value of the firm).
Finally, following a structural approach, we define the default of the company as

$$\tau := \inf\{t \geq 0 : V_t \leq a\}, \tag{0.3}$$

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for a given real constant parameter a such that $0 < a < v_0$ and where, as usual, $\inf \emptyset = +\infty$.

We consider three different filtrations, satisfying the usual hypotheses, representing three different levels of information available to agents in the market, namely, for each $t \geq 0$,

$$\mathcal{F}_t^S := \sigma(S_s, 0 \leq s \leq t), \quad \mathcal{F}_t^{S,\tau} := \mathcal{F}_t^S \vee \sigma(t \wedge \tau) \quad \text{and} \quad \mathcal{G}_t.$$

$(\mathcal{F}_t^{S,\tau})_{t \geq 0}$ is the information to which intuitively the investor has access, while $(\mathcal{G}_t)_{t \geq 0}$ is the full information filtration, i.e., the information available for example to a small number of stock holders of the company, who have access to S and V at each time t. In our case, for example, the full filtration is the one generated by the stochastic pair process $(W, \bar{W})$ and finally we have

$$\mathcal{F}_t^S \subset \mathcal{F}_t^{S,\tau} \subset \mathcal{G}_t, \quad \forall t \geq 0.$$

Our aim is to compute, for each $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $s \leq t \leq T$ (i.e., we have fixed a time horizon), the following key quantity, which is fundamental when pricing credit derivatives under partial information

$$\mathbb{P}\left(\tau > t | \mathcal{F}_s^{S,\tau}\right) = \mathbb{P}\left(\inf_{s \leq u \leq t} V_u > a \middle| \mathcal{F}_s^{S,\tau}\right). \tag{0.4}$$

By simply using the law of iterated conditional expectations, we find

$$\begin{aligned}
\mathbb{P}\left(\tau > t | \mathcal{F}_s^{S,\tau}\right) &= \mathbb{E}\left\{\left[\mathbb{1}_{\{\tau \leq s\}} + \mathbb{1}_{\{\tau > s\}}\right] \mathbb{P}\left(\inf_{0 \leq u \leq t} V_u > a \middle| \mathcal{G}_s\right) \middle| \mathcal{F}_s^{S,\tau}\right\} \\
&= \mathbb{1}_{\{\tau > s\}} \mathbb{E}\left[\mathbb{P}\left(\inf_{s \leq u \leq t} V_u > a \middle| \mathcal{G}_s\right) \middle| \mathcal{F}_s^{S,\tau}\right] \\
&= \mathbb{1}_{\{\tau > s\}} \mathbb{E}\left[\mathbb{P}\left(\inf_{s \leq u \leq t} V_u > a \middle| V_s\right) \middle| \mathcal{F}_s^{S,\tau}\right] \\
&= \mathbb{1}_{\{\tau > s\}} \mathbb{E}\left[F(s,t,V_s) | \mathcal{F}_s^{S,\tau}\right],
\end{aligned} \tag{0.5}$$

so that,

(P1) *if* we compute F for each pair (s,t) and

(P2) *if* we obtain the *filter distribution*, i.e., the conditional distribution of V_t given $\mathcal{F}_t^{S,\tau}$, $\Pi_{V_t | F_t^{S,\tau}}$, for each $0 \leq t \leq T$,

then we are done, since it suffices to compute the integral

$$\mathbb{1}_{\{\tau > s\}} \mathbb{E}\left[F(s,t,V_s) | \mathcal{F}_s^{S,\tau}\right] = \mathbb{1}_{\{\tau > s\}} \int_a^\infty F(s,t,x) \Pi_{V_s | F_s^{S,\tau}}(\mathrm{d}x).$$

We show how to numerically solve (**P1**) by means of Monte-Carlo simulations of process $(\tilde{V}_t)_t$, the continuous version of the Euler scheme relative to V and we obtain the discrete time approximation of $\Pi_{V_t | F_t^{S,\tau}}$ by optimally quantizing our signal V, inspired by [3].
Our results are comparable with the ones in [2] and in [1], so that, due to the flexible applicability of the quantization procedure, we can analyze and compare the spread curves under complete and partial information in new and more general settings.

References

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