Economic Capital and Optimal Investment for Non-Life Insurance Companies

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Abstract

We propose two optimization models to jointly solve a capital requirement and a portfolio selection problem for non-life insurance companies. In a one-period framework, the expected return on risk-adjusted capital (RORAC) is optimized subject to zero-conditional value at risk shortfall constraint. Equity capital adjustment to match the global balance sheet economic capital is assumed to be exclusively allocated in a risk-free asset. A closed-form relationship is derived between optimal economic capital and investment portfolio. Using a Monte Carlo simulation calibrated to major French non-life insurance company data, we develop first a curve of efficient portfolios verifying the shortfall constraint. Then, we analyze optimum sensitivity to some factors. It turns out that, depending on adjustment costs, risk management by equity shift can be more effective than assets risk management with a fixed capital. Furthermore, optimum presents a high sensitivity to initial surplus level and dependence structure between assets and liabilities. The numerical results are robust to the relaxation of the assumption regarding equity adjustment allocation.

keywords: Portfolio selection, Capital management, Risk-based economic capital, Stochastic fractional programming, Non-life insurance.

JEL-Classification: G11, G28 & G32

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1. Introduction

There is an increasing concern about quantifying capital requirement for financial companies. Actually, the major part of insurance regulation advance aims to provide a clear guideline for insurers to deal with this issue (see, Eling et al. (2007, 2008)). For example, the new European insurance regulatory system Solvency II points particularly in this direction. In this risk based capital system, required capital is based on two building blocs: the minimum capital requirement and the solvency capital requirement very close to the theoretical concept of economic capital.

An extensive research has focused on quantifying economic capital for insurance firms under some assumptions on distributions or dependence structure of underlying risks (see, e.g., Djehiche and Hörflet (2004), Tang and Valdez (2005), Dhaene et al. (2006) and Bisignani et al. (2007)). These studies concentrate mainly on the conventional view of capital as "buffer stock". However, as described in Dvorak (2007), capital could be also seen as an essential input to the firm growth, for instance, by launching new business, undertaking M&A or, more commonly, having more access to financial markets. Berger et al. (2008) find evidence in favour of a high capital ratio and an active managed capital hypotheses for publicly traded U.S. bank holding companies[1]. A prominent explanation of these findings is the fact that holding more capital increases the company's capacity for taking further risks and increasing profit. However, excessive equity capital reduces competitiveness because of high equity cost and some market frictions, Cummins and Nini (2002). This trade off highlights the key role of economic capital in risk management and investment decisions.

On the other hand, optimal financial investment for insurance firms has covered a large body of literature. Besides, the importance of investment revenue in the net income, recent regulation changes regarding financial investment possibilities for insurers renew interest for this issue. At the best of our knowledge, this field of research starts with the works of Kahane and Nye (1975) and Cummins et al. (1981) for single period model. Browne (1995) extends the problem into a dynamic framework. Without a budget constraint, he shows that optimal strategy consists in holding a constant amount of funds in the risky asset independently of the surplus size. On contrary, Hipp and Plum (2000) find that optimal amount invested in the risky asset is a function of the current surplus. This difference in results is due to the model characterization. Further extensions have been considered in literature. Among others, Liu and Yang (2004) generalize Hipp and Blum (2000) model by including a risk-free asset. Luo et al. (2008) consider the same problem when neither short selling nor borrowing are allowed. These studies follow mainly two objectives maximizing the expected exponential utility or minimizing the probability of ruin. In both cases, capital is viewed as an exogenous state variable.

For all the above literature, optimization process concerns only one control variable either assets allocation or economic capital. It is increasingly admitted in financial theory that risk management and investment decisions are interdependent (e.g., Froot, Scharfstein, and Stein (1993), Froot and Stein (1998), Leland (1998), Mello and Parsons (2000), Peura and Keppo (2006) and Lin et al. (2008)).

[1] One could expect comparable results for insurance firms to the tested hypothesis of Berger et al. (2008). The variables used to explain the ratio of capital such as market-to-book ratio, acquisition opportunities and firm's size remain appropriate for the insurance context.

Therefore, optimizing with respect to only one control variable is likely to be suboptimal. A general optimum is found by a joint optimization as suggested by Ahcan et al. (2004) Dhaene et al. (2005) and Planchet and Therond (2007).

In this paper, we propose two optimization models to solve a jointly assets allocation/capital requirement problem for non-life insurance firms. Under dependent assets and liabilities risks assumption, we optimize in a one-period model a fractional objective function return on risk-adjusted capital (RORAC) subject to a zero-conditional value at risk shortfall constraint.CVaR metric exhibits two interesting properties for applications in finance and insurance, namely convexity and subadditivity. Moreiver, because of its attractive computational features It is considered by academicians and practitioners as an alternative to the Value at risk metric. Independence between assets and liabilities risks has long been adopted in actuarial studies. However, such assumption appears to be in contradiction with some stylized facts reported in the literature, see Denuit and Scaillet (2004). Equity adjustment to match optimal economic capital is assumed to be exclusively allocated in a risk-free asset.

We derive a closed-form relationship between economic capital and optimal portfolio. Using a Monte Carlo simulation based on of major French non-life insurance company data, we develop a curve of efficient portfolios verifying the shortfall constraint. Then, we analyze and evaluate optimum sensitivity. It turns out that, depending on capital adjustment costs, risk management by equity adjustment can be more effective than assets risk management with a fixed capital. Furthermore, optimum presents a high sensitivity to initial capital level and dependence structure between assets and liabilities risks. We numerically demonstrate equivalence of results to the relaxation of the assumption regarding the allocation of equity capital adjustment.

The remainder of this paper is structured as follows. Section 2 sets out the analytical framework and provides a solution to the non-life insurer problem without a solvency constraint. Section 3 proposes two equivalent approaches to solve the problem with a solvency constraint. In Section 4, we introduce the case study to test our theoretical results. In Section 5, we discuss various results connected with optimum characteristics and sensitivity to some factors. Finally, section 6 concludes.

2 Problem solution without a shortfall constraint

Before giving the general solution, we formulate and solve in this section the problem in a general setting and present some theoretical results. The latter are similar to those obtained from the portfolio selection problem under a conditional value at risk constraint and fixed capital. The motivation behind this intermediate step is to use the related developments in later sections.

2.1 Notations and assumptions

We consider for a single period[2] an unlevered non-life insurance firm starting with (I) lines of business indexed by $i = 1$, 2, ..., I. The firm receives deterministic premium incomes $\mathbf{p}= (p_1, \ldots, p_I)^T$ at the beginning of period and pays contingent claims $\widetilde{\mathbf{s}}=(\widetilde{s}_1, \ldots, \widetilde{s}_I)^T$ at the end of the period. It does not

[2] In practice, the period is one year long. We could apply our model for less short time horizon.

offer new contracts meanwhile. The liabilities risk cannot be transferred nor hedged in the marketplace[3]. At the beginning of period, the firm holds technical provisions P_t and starts with an initial capital C_t assumed to be superior to some minimum capital requirement C_0[4]. Available funds $P_t + C_t$ are entirely invested following a strategic asset allocation in $(J+1)$ financial assets $(A_0, \ldots, A_J)$ with random holding period return rates $\widetilde{\mathbf{r}} = (r_0, \widetilde{r}_1, \ldots, \widetilde{r}_J)^T$ sorted in an increasing order $(r_0 \prec E(\widetilde{r}_1), \ldots \prec E(\widetilde{r}_J))$. We assume that return rates and claims have a continuous joint distribution. Let $\mathbf{x} = (x_1, \ldots, x_J)^T$ be the assets portfolio weights, such that $x_j \geq 0$ i.e. (short sales are disallowed) and $x_0 = 1 - \mathbf{x}^T e$ (budget constraint). We express the basic portfolio restrictions in vector notation as: $X = \{\mathbf{x} \in [0,1]^J \mid \mathbf{x}^T e \leq 1\}$. Throughout the paper we use boldface characters to denote vectors. We consider the following notation:

- $\widetilde{L}$ loss function
- $\widetilde{s}_i$ agregate claim costs associated to the i^{th} line of business
- C_t capital level at the beginning of the period
- P_t technical provisions at the beginning of the period
- r_0 holding period rate of return on the riskless asset
- $\widetilde{r}_j$ holding period rate of return on j^{th} risky asset
- x_j proportion of wealth invested in j^{th} risky asset
- p_i net premium income for the i^{th} line of business in the period
- τ tax rate

We define a linear loss (disutility) function as equal to the difference between aggregate claims and assets value at the end of period.

$$\widetilde{L}(\mathbf{x}, \widetilde{\mathbf{s}}, \widetilde{\mathbf{r}}) = \widetilde{\mathbf{s}}^T \mathbf{i} - (P_t + C_t)\left[(1 + r_0) + \mathbf{x}^T(\widetilde{\mathbf{r}} - r_0 \mathbf{e})\right] \quad (1)$$

where $\mathbf{e} \in R^J$ and $\mathbf{i} \in R^I$ are J-dimensional and I-dimensional unit vector $(1, 1, ..., 1)^T$. To facilitate the analysis, we assume that $(L \succ 0)$ imply a loss. This function enters in the definition of the conditional value at risk. As introduced by Rockafellar and Uryasev (2000), for continuous distributions, CVaR is equal to:

$$CVaR_\alpha(\widetilde{L}) = E(\widetilde{L}/\widetilde{L} \geq VaR_\alpha)$$

It is defined, for a given confidence level α, as the conditional expectation of loss function exceeding or equal to value at risk (VaR). The latter concept is defined as a percentile of a loss function:

$$VaR_\alpha(\widetilde{L}) = \inf\left\{l \in \mathbb{R} \mid P(\widetilde{L} \succ l) \prec \alpha\right\}$$

Conditional value at risk verifies coherence and convexity properties introduced by Artzner et al. (1999) and Föllmer and Schied (2002). A coherent risk measure $\rho(X)$ of an investment with the random

[3] Without risk management, insurance firms have two choices to verify regulation constraints. The first one consists on reducing aggregate risk by changing asset's structure towards more liquid and risk-free assets (cash). The second alternative is to increase their risk-bearing capacity by holding more equity capital, Peura and Keppo (2006).

[4] Solvency II defines the minimum capital requirement (MCR) as the final threshold of capital that could trigger ultimate supervisory measures in the case that it is breached.

return X is a real-valued function defined on the space of real-valued random variables and satisfying the following axioms:

(i) (Translation invariance) For any $a \in R$, $\rho(X-a)=\rho(X)-a$.

(ii) (Subadditivity). For any random variables X and Y, $\rho(X+Y)=\rho(X)+\rho(Y)$.

(iii) (Positive homogeneity). For all $t \geq 0$, $\rho(tX)=t\rho(X)$.

(iv) (Monotonicity) if X and Y for each outcome, then $\rho(X) \geq (Y)$.

A convex risk measure satisfies axioms (i)-(iii) and also:

(v) (Convexity) $\forall\lambda \in [0,1]$; $\rho(\lambda X+ (1-\lambda)\, Y) \leq \lambda\rho(X)+ (1-\lambda)\, \rho\,(Y)$.

One of the major drawbacks of VaR is the failure to verify axioms (ii) and (v) for general distributions of risk. The translation invariance property of $CVaR$ may be interpreted as the amount of cash to add to the risky position(withdraw) when $CVaR(X) \geq 0$ ($CVaR(X) \leq 0$) to completely cancel the risk. For the optimization process, we will use an alternative definition of $CVaR$. As showed by Rockafellar and Uryasev (2000) optimizing $CVaR$ with respect to $\mathbf{x}$ is equivalent to optimize some specific function $F_{\alpha}\,(x,u)$.

$$\min_{\mathbf{x}} CVaR\,(\mathbf{x}) = \min_{\mathbf{x}\in X,\, u\in\mathbb{R}} F_{\alpha}\,(\mathbf{x},u) \tag{2}$$

with,

$$F_{\alpha}\,(\mathbf{x},u) = u + \tfrac{1}{\alpha}E\left\{\left[\widetilde{L}\,(\mathbf{x},\widetilde{\mathbf{s}},\widetilde{r}) - u\right]^{+}\right\},\, u \in \mathbb{R}.$$

The advantage of this formulation is to transform the optimal portfolio problem into a linear program which is efficiently solvable. We also assume that $E\left[\left|\widetilde{L}\,(\mathbf{x},\widetilde{\mathbf{s}},\widetilde{r})\right|\right] \prec +\infty$ for each $\mathbf{x} \in X$, so that conditional value at risk will be properly defined[5].

The overall objective of the insurer is to maximize a fractional objective function, namely, the return on risk-adjusted capital (RORAC). Matten (2000) defines this index as the ratio between the income and economic capital EC_t:

$$RORAC^0 = \frac{\left[(\mathbf{p}-\widetilde{\mathbf{s}})^T\,\mathbf{i} + (P_t + C_t)\left[(r_0 + \mathbf{x}^T\,(\widetilde{\mathbf{r}} - r_0\mathbf{e})\right]\right](1-\tau)}{EC_t} \tag{3}$$

Other typical examples of objective functions used in the literature include ruin probability and exponential utility of terminal surplus. Planchet and Therond (2007) emphasize the relevence of using a ratio criterion such as profitability or surplus to solvency or economic capital. The RORAC responds to regulator and stockholders requirements since it takes into consideration profitability and risk.

2.2 Optimal portfolio under a general CVaR constraint

We maximize first the return on capital ROC based on a general CVaR risk constraint. ROC is defined as the ratio between the net income and the fixed initial capital level C_t. We determine then the optimal portfolio for a maximal value of $CVaR_{\alpha}^{0}$. One has to solve this problem:

[5]Bamberg and Neuhierl (2008) point out that short positions and heavy tail distributions of risks are incompatible with a finite CVaR.

$$[P_\alpha^0]\ \ \phi(V_\alpha) = \underset{\mathbf{x}\in X}{Max} E\left[ROC^0(\mathbf{x})\right]$$
$$S.t\ \ CVaR_\alpha^0(\mathbf{x}) \leq V_\alpha$$

An alternative formulation consists on minimizing $CVaR_\alpha^0$ for a given minimal level of expected return as in Rockafellar and Uryasev (2002):

$$[Q_\alpha^0]\ \ \psi(E_\alpha)\quad \underset{\mathbf{x}\in X}{Min}\, CVaR_\alpha^0(\mathbf{x})$$
$$S.t\ \ E\left[ROC^0(\mathbf{x})\right] \geq E_\alpha$$

Where E_α and V_α are the expected return rate and risk limits. As proved by Krokhmal et al. (2002), problems $[P_\alpha^0]$ and $[Q_\alpha^0]$ lead to the same efficient frontier obtained by varying E_α and V_α, respectively. We refer to the theoretical results obtained by Li et al. (2001). They focused on optimal portfolio selection of assets with transaction costs and no short sales. The framework used in their study can be adapted for the non-life insurer optimal portfolio selection in the case whit fixed capital. Actually, the feasible set X defined above is compact and convex, ROC^0 is concave (linear) and $CVaR_\alpha^0$ is convex. An important result states that the constraints $CVaR_\alpha^0 \leq V_\alpha$ for $[P_\alpha^0]$ and $ROC^0 \geq E_\alpha$ for $[Q_\alpha^0]$ bind when $V_\alpha \in \left[V_{\alpha,\min}, V_{\alpha,\max}\right]$ and $E_\alpha \in \left[E_{\alpha,\min}^{(Q)}, E_{\alpha,\max}^{(P)}\right]$, respectively, where $(E_{\alpha,\max}^{(P)}, V_{\alpha,\max}^{(P)})$ and $(V_{\alpha,\min}^{(Q)}, E_{\alpha,\min}^{(Q)})$ correspond to the (respective) solutions of the unconstrained problems $[P_\alpha^{uc,0}]$ and $[Q_\alpha^{uc,0}]$:

$$[P_\alpha^{uc,0}]:\quad \underset{\mathbf{x}\in X}{Max}\quad E\left[ROC^0(\mathbf{x})\right] \qquad\qquad [Q_\alpha^{uc,0}]:\quad \underset{\mathbf{x}\in X}{Min}\quad CVaR_\alpha^0(\mathbf{x})$$

It was well documented in numerous studies concerning portfolio selection under a CVaR constraint, that efficient frontier $\phi(V_\alpha) = E\left[ROC^0(V_\alpha)\right]$ is concave and strictly increasing. We refer to Krokhmal et al. (2002) and Bertsimas et al. (2004) results which still remain true for the case of an insurer with a fixed capital.

Note that depending on the value of $CVaR^0\left[\mathbf{x}^*(V_{\alpha,\min})\right]$[6] and $ROC^0\left[\mathbf{x}^*(E_{\alpha,\max})\right]$, we distinguish three cases regarding efficient frontier position. The first case is related to a situation of distress typically under-capitalization. Even for the minimum risk portfolio, the firm is not able to carry out solvency constraint. Conversely, the second case describes an overcapitalized firm for which all efficient portfolios have negative risk. The last and the most interesting case concerns a fair-capitalized insurer which fulfils the following condition: $CVaR^0\left[\mathbf{x}^*(V_{\alpha,\min})\right] \prec 0$ and $ROC^0\left[\mathbf{x}^*(E_{\alpha,\max})\right] \succ 0$. In the sequel, we will exclusively concentrate on the last case.

3. Problem solution under a shortfall constraint

In this section, we solve the portfolio selection problem under a zero-CVaR shortfall constraint of the form:

$$CVaR_\alpha\left[\widetilde{L}(\mathbf{x},\widetilde{\mathbf{s}},\widetilde{\mathbf{r}},\Delta C_t)\right] = 0$$

Where ΔC_t is the adjusted equity capital, invested at the risk free rate i.e. considered as buffer cap-

[6] It deonotes the aggregate risk of the firm when investing all its funds on a risk-free asset (when it corresponds to the minimum risk portfolio) i.e facing only insurance risk.

ital[7]. Equity capital adjustment involves brokerage commissions. We specify a symmetric cost function $\delta\left(\Delta C_t\right) = b\left|\Delta C_t\right|$, $b \succ 0$. To derive optimal solution, we propose two methodologies: the two-step approach and the fractional approach.

3.1 The two-step approach

The first step consists on deriving the optimal portfolio $\mathbf{x}^*_{V_\alpha}$ from $\left[P^0_\alpha\right]$. Since the risk constraint in $\left[P^0_\alpha\right]$ is binding and relying on the translation invariance property of $CVaR$, we have:

$$\forall V_\alpha \in \left[V_{\alpha,\min}, V_{\alpha,\max}\right] \quad CVaR^0_\alpha\left[L\left(\mathbf{x}^*_{V_\alpha}\right) - V_\alpha\right] = 0$$

The risk tolerance level V_α can be viewed as the amount of numeraire to add to the loss function for an efficient portfolio to cancel the aggregate risk. Numeraire comes from raised equity capital ΔC_t allocated in a risk free asset. There is an exchangeability relationship between V_α and ΔC_t such that:

$$\Delta C_t = \frac{V_\alpha}{1 + r_0}$$

In the second step, we optimize with respect to V_α the RORAC function related to optimal portfolios. After reformulation, one has to optimize the following problem:

$$\left[P^1_\alpha\right] \max_{V_\alpha \in \left[V_{\alpha,\min}, V_{\alpha,\max}\right]} E[RORAC(\mathbf{x}^*\left(V_\alpha\right))] = C_t \frac{\left[ROC^0\left(\mathbf{x}^*\left(V_\alpha\right)\right) - h\right]}{C_t + \frac{V_\alpha}{(1+r_0)}} + h(1-\tau) \qquad (4)$$

Where,

$$\begin{cases} h = r_0 - b & \text{if } \Delta C_t \geq 0 \\ h = r_0 + b & \text{if } \Delta C_t \leq 0 \end{cases} \qquad (5)$$

[7] This assumption is very restrictive, it allows however to keep constant the value of investable funds. It was adopted to obtain a closed form for optimal adjusted capital ΔC_t.

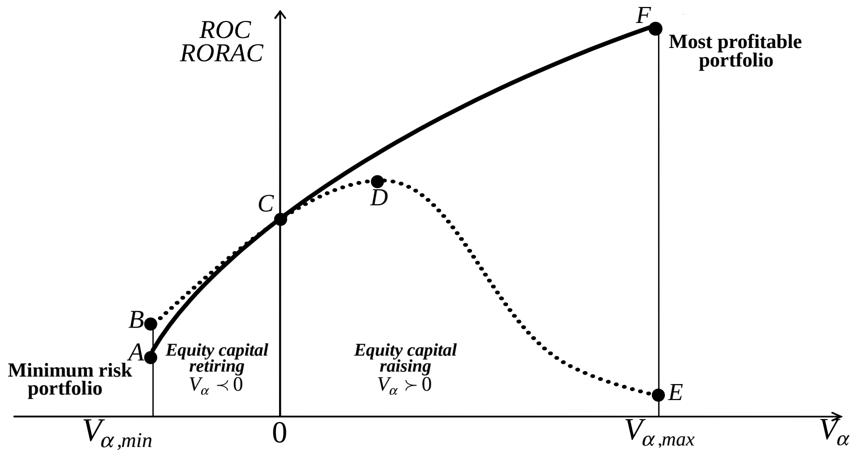

Fig. 1. Efficient frontier curve (plain line) of $ROC^0(V_\alpha)$ without a solvency constraint and $RORAC[\mathbf{x}^*(V_\alpha)]$ curve (dashed line) of efficient portfolios.

As an illustration, figure 1 depicts the efficient frontier $ROC^0\left(V_\alpha\right)$ and the $RORAC\left(V_\alpha\right)$ curve for optimal portfolios verifying the solvency constraint. Efficient frontier associated to problem $\left[P_\alpha^0\right]$ (plain line) cuts the Y-axis at point C where $V_\alpha = 0$. At this point, the solvency constraint binds without need for capital adjusting. At the left side of Y-axis, $V_\alpha \prec 0$ i.e. with initial equity capital C_0 the risk is negative (overcapitalization). At the right side of Y-axis, $V_\alpha \succ 0$ i.e. with initial equity capital C_0 the risk is positive and without raising equity, solvency constraint will not be verified. The points A and F are the bounds of efficient frontier. We assume that $\forall V_\alpha \in \left[V_{\alpha,\min}, V_{\alpha,\max}\right]$ the economic capital $C_t + \frac{V_\alpha}{(1+r_0)}$ is nonnegative. The RORAC objective function is strictly quasi-concave on $\left[V_{\alpha,\min}, V_{\alpha,\max}\right]$ since it corresponds to a ratio of nonnegative concave and linear functions. Its associate curve (dashed line) is expected to admit a global maximum (point D). At the minimum risk portfolio $\mathbf{x}^*_{V_{\alpha,\min}}$, if the firm retires equity capital by $\frac{\left|V_{\alpha,\min}\right|}{(1+r_0)}$, the RORAC will move from A to B. At the most profitable portfolio $\mathbf{x}^*_{V_{\alpha,\max}}$, RORAC is low. To carry out the solvency constraint for this risky investment, the firm has to increase equity capital with a high cost. Consequently, RORAC declines and moves from F to E. It is important to note that adjusting (raising) equity capital improves the objective function value which moves from C to D. As long as the maximum of the RORAC curve is located at the right of Y-axis, it will be profitable for the firm to raise equity capital. At the left of Y-axis, to improve RORAC by retiring equity capital, the return rate at the point D should be superior to the one at point C[8]. Otherwise, keeping unchanged initial capital would be the optimal solution.

Remark

The RORAC curve is designed as an illustration of its shape. The position of this curve could be shifted more to the left or to the right depending on the value of parameters. There are two extreme cases concerning this curve. The first case is when (D=A) i.e. the optimal portfolio overlaps the minimum risky portfolio of $\left[P_\alpha^0\right]$. An important diversification effect within and between investment and insurance

[8] The position of point D to the point C depends on the adjusting equity capital cost function.

risks could explain this situation. Consequently, the RORAC curve will be convex and decreasing. The second case is when (D=F) i.e. the optimal portfolio overlaps the most profitable portfolio of $\left[P_\alpha^0\right]$. A possible explanation of this situation will be an excessive negative insurance income. The RORAC curve will be concave and increasing[9].

3.1 The fractional approach

To calculate the optimal portfolio based on RORAC criterion, one has to solve the following fractional program:

$$\begin{array}{l} [P_\alpha^2] \underset{(\mathbf{x},\Delta C_t)}{Max}\, E\left[RORAC\right] = \frac{\left[(\mathbf{p}-E(\widetilde{\mathbf{s}}))^T\mathbf{i}+(P_t+\,C_t)\left[(r_0+\mathbf{x}^T(E(\widetilde{\mathbf{r}})-r_0\mathbf{e})\right]+r_0\Delta C_t-\delta(\Delta C_t)\right](1-\tau)}{C_t+\Delta C_t} \\ S.t\;\; CVaR_\alpha\left[\widetilde{\mathbf{s}}^T\mathbf{i}-(P_t+\,C_t)\left[(1+r_0)+\mathbf{x}^T\left(\widetilde{\mathbf{r}}-r_0\mathbf{e}\right)\right]-\Delta C_t(1+r_0)\right]\leq 0 \\ \mathbf{x}\in X \end{array} \tag{6}$$

We next show that $CVaR$ risk constraint binds.

Proposition 1 *The risk constraint of the optimization problem* $\left[P_\alpha^2\right]$ *binds at the optimal portfolio.*
Proof. *Let us suppose, on the contrary, that optimal solutions* $(\mathbf{x}^*, \Delta C_t^*)$ *verify:*

$$CVaR_\alpha\left[\widetilde{\mathbf{s}}^T\mathbf{i}-(P_t+\,C_t)\left[(1+r_0)+\mathbf{x}^{*T}\left(\widetilde{\mathbf{r}}-r_0\mathbf{e}\right)\right]-\Delta C_t^*(1+r_0)\right]\prec 0$$

Using, translation invariance property of CVaR:

$$CVaR_\alpha^0\left[\widetilde{\mathbf{s}}^T\mathbf{i}-(P_t+\,C_t)\left[(1+r_0)+\mathbf{x}^{*T}\left(\widetilde{\mathbf{r}}-r_0\mathbf{e}\right)\right]\right]\prec \Delta C_t^*(1+r_0)$$

where $CVaR_\alpha^0(x^*)$ *is the risk measure computed for the optimal allocation* x^**when capital is fixed. There exists an adjusted capital* $\left[\widetilde{\Delta C}_t\in\mathbb{R}\mid\widetilde{\Delta C}_t\prec\Delta C_t^*\right]$ *satisfying the risk constraint such that:*

$$CVaR_\alpha^0\left[\widetilde{\mathbf{s}}^T\mathbf{i}-(P_t+\,C_t)\left[(1+r_0)+\mathbf{x}^{*T}\left(\widetilde{\mathbf{r}}-r_0\mathbf{e}\right)\right]\right]=\widetilde{\Delta C}_t\,(1+r_0)$$

But, since the expected $RORAC$ *can be expressed as:*

$$E\left[RORAC\right]=\frac{\left[(\mathbf{p}-E(\widetilde{\mathbf{s}}))^T\mathbf{i}+(P_t+\,C_t)\left[(r_0+\mathbf{x}^T(E(\widetilde{\mathbf{r}})-r_0\mathbf{e})\right]-hC_t\right](1-\tau)}{C_t+\Delta C_t}+h(1-\tau)$$

$$=\frac{\left[(\mathbf{p}-E(\widetilde{\mathbf{s}}))^T\mathbf{i}+P_t\left[(r_0+\mathbf{x}^T(\widetilde{\mathbf{r}}-r_0\mathbf{e})\right]+\,C_t\left[\mathbf{x}^T(E(\widetilde{\mathbf{r}})-r_0\mathbf{e})\right]\pm bC_t\right](1-\tau)}{C_t+\Delta C_t}+h(1-\tau)$$

$\forall\, x^*\in X$:

$$E[RORAC(x^*,\,\widetilde{\Delta C}_t)]\succ E[RORAC(x^*,\Delta C_t^*)]$$

As we assume that :

$$(\mathbf{p}-E\left(\widetilde{\mathbf{s}}\right))^T i+P_t\left[(r_0+\mathbf{x}^T\left(\widetilde{\mathbf{r}}-r_0\mathbf{e}\right)\right]+\;C_t\left[r_0+\mathbf{x}^T\left(E\left(\widetilde{\mathbf{r}}\right)-r_0\mathbf{e}\right)\right]\pm bC_t\geq 0$$

and $C_t+\Delta C_t\geq 0$

$\left(\mathbf{x}^*,\widetilde{\Delta C}_t\right)$ *is a feasible solution for the problem above and improves the optimum given by* $(\mathbf{x}^*, \Delta C_t^*)$*; which contradicts the optimality of* $(\mathbf{x}^*, \Delta C_t^*)$. ■

[9]We test numerically these two extreme cases.

The general formula of the adjusted equity capital is shown in the following proposition whose proof is straightforward from the last result.

Proposition 2 *Optimal adjusted equity capital is equivalent to:*

$$\Delta C_t^* = \frac{CVaR_\alpha^0(\mathbf{x}^*)}{(1+r_0)} \tag{7}$$

where $CVaR_\alpha^0(x^*)$ is the risk measure computed for the optimal allocation $x^*$when capital is fixed.
Proof. *Since the risk constraint is binding and using translation invariance property of CVaR, we have: $\Delta C_t = \frac{CVaR_\alpha^0(x)}{1+r_0}$.Then, after reformulation of $\left[P_\alpha^2\right]$, one has to solve this problem:*

$$\underset{\mathbf{x}\in X}{max} E[RORAC(\mathbf{x})] = C_t \frac{\left[ROC^0(\mathbf{x}) - h\right]}{C_t + \frac{CVaR^0(\mathbf{x})}{(1+r_0)}} + h(1-\tau) \tag{8}$$

Where, h is the same as in (5). The feasible set is compact convex and the objective function is a strictly quasi-concave[10] since the numerator $N(x)$ is affine and nonnegative and the denominator $D(x)$ is convex (due to the convexity of CVaR). Therefore, the maximization problem $\left[P_\alpha^2\right]$ is said to be a concave convex fractional program and admits a unique solution x^* so that:

$$\Delta C_t^* = \frac{CVaR_\alpha^0(\mathbf{x}^*)}{1+r_0} \quad \blacksquare$$

To solve the fractional problem $N[\mathbf{x}] \,/\, D[\mathbf{x}]$, we use the Dinkelbach parametric approach. This method is based on solving numerically an auxiliary problem $Q(\mathbf{x},\lambda) = N[\mathbf{x}] - \lambda D[\mathbf{x}]$, N and D refer to the numerator, denominator and a parameterλ. As described in Schaible (1976) and Philips (1998), the Dinkelbach algorithm is based on four steps:

Step 1 (Initialization).Selection some vector$\left[x_1^{(0)}, \cdots x_m^{(0)}\right] \in X$ and setting $\lambda^{(k=0)} = \frac{N\left[x_1^{(0)}, \cdots x_m^{(0)}\right]}{D\left[x_1^{(0)}, \cdots x_m^{(0)}\right]}$.

Step 2 (Solution search). Solving the problem $Q(\lambda^0)$ to get the optimal solution point $\left[x_1^{(k+1)}, \cdots x_m^{(k+1)}\right]$.

Step 3 (Stopping test). If $Q(\lambda) = N\left[x_1^{(k+1)}, \cdots x_m^{(k+1)}\right] - \lambda^{(k)} D\left[x_1^{(k+1)}, \cdots x_m^{(k+1)}\right] = 0$, then setting $[x_1^*, \cdots x_m^*] = \left[x_1^{(k+1)}, \cdots x_m^{(k+1)}\right]$, $\lambda^* = \lambda^{(k)}$, and stopping.

Step 4 (Loop). If $Q(\lambda) = N\left[x_1^{k+1}, \cdots x_m^{k+1}\right] - \lambda^k D\left[x_1^{k+1}, \cdots x_m^{k+1}\right] > 0$, then setting $\lambda^{k+1} = \frac{N\left[x_1^{k+1}, \cdots x_m^{k+1}\right]}{D\left[x_1^{k+1}, \cdots x_m^{k+1}\right]}$, $k = k+1$ and going to step 2.

Once the optimal portfolio $\mathbf{x}^*$ is obtained, it is straightforward to derive the optimal adjusted capital ΔC_t^* from(7).

Proposition 3 *The two optimization problems $[P_\alpha^1]$ and $[P_\alpha^2]$ are equivalent in the sense that they produce the same optimal portfolio.*

[10]For an exhaustive review of fractional programming literature refer to the survey papers of Schaible (1995) and Frenk and Schaible (2009).

Proof. Let assume on contrary that optimal portfolio $\mathbf{y}^* \in X$ in the problem $\left[P_\alpha^2\right]$ does not belong to set of optimal solutions $\mathbf{x}_{V_\alpha}^* \in X$ of $\left[P_\alpha^0\right]$. Then, $ROC^0\left(\mathbf{y}^*\right) \leq ROC^0\left(\mathbf{x}_{V_\alpha}^*\right)$, $CVaR_\alpha^0\left(\mathbf{y}^*\right) \geq CVaR_\alpha^0\left(\mathbf{x}_{V_\alpha}^*\right)$, where at least one the inequalities is strict. For instance, $ROC^0\left(\mathbf{y}^*\right) \prec ROC^0\left(\mathbf{x}_{V_\alpha}^*\right)$ and $CVaR_\alpha^0\left(\mathbf{y}^*\right) = CVaR_\alpha^0\left(\mathbf{x}_{V_\alpha}^*\right)$. Since $C_t + CVaR_\alpha^0\left(\mathbf{y}^*\right) \succ 0$, it follows that:

$$\frac{ROC^0\left(\mathbf{y}^*\right)}{C_t + CVaR_\alpha^0\left(\mathbf{y}^*\right)} \prec \frac{ROC^0\left(\mathbf{x}_{V_\alpha}^*\right)}{C_t + CVaR_\alpha^0\left(\mathbf{x}_{V_\alpha}^*\right)} \tag{9}$$

Which contradicts the optimality of $\mathbf{y}^*$. Then the optimal solution of the problem $\left[P_\alpha^2\right]$ belongs to the set of optimal solutions $\mathbf{x}_{V_\alpha}^*$ of $\left[P_\alpha^1\right]$. Furthermore, $\left[P_\alpha^1\right]$ and $\left[P_\alpha^2\right]$ admit unique solutions and since the risk constraint in $\left[P_\alpha^0\right]$ is binding, it follows that:

$$\mathbf{y}^* = \mathbf{x}_{V_\alpha^*}^* \qquad where \quad V_\alpha^* = CVaR_\alpha^0\left(\mathbf{y}^*\right) \tag{10}$$

Therefore, $\left[P_\alpha^1\right]$ and $\left[P_\alpha^2\right]$ generate the same optimal portfolio. ■

Remark

Without further assumptions on the distribution of return rates and claims, it is not possible to advance beyond the above results. If risks have a joint elliptic distribution, where CVaR has a closed-form expression, we can extend the initial problem to find, for instance, the conditions on parameters of the joint distribution to verify $\Delta C_t^* \geq 0$ or $\Delta C_t^* \leq 0$.

Perhaps the most complicated extension concerns the implications on optimal solutions of relaxing the assumption of the reserve role of adjusting capital. In a more general setting, this capital will be allocated optimally between the different assets. Therefore, one has to show whether there exists an in non convex problem an optimal portfolio improving the RORAC compared to the basis problem solution. Depending on the joint distribution parameters, there may exist such optimal portfolio. However, because of capital adjustment cost, it is expected to this new problem leads to the same solution to the basic one.

4. Case study and numerical simulation

We investigate the robustness of our theoretical results based on data from a major French non-life insurance firm. To create a realistic representation of the industry, data cover four principal lines of business (LoB) in terms of gross paid claims over the period 1995-2006[11]. The four business lines selected are: motor, third-party, fire and property damage. Because of the small number of observations, we use bootstrap method to compute the first two moments and correlation between the four LoB. Table 1 presents a summary of aggregate claims descriptive statistics.

[11] The lack of data prevents us from using a larger sample. French insurance companies' data have begun to be officially reported in 1995.

Table1. Claims summary statistics of the first two moments and correlation

	Motor	Third-party	Fire	Property damage
Motor	1	0.09 (0.000)*	0.227 (0.26)	-0.202 (0.12)
Third-party	-	1	-0.415 (0.11)	0.099 (0.39)
Fire	-	-	1	0.25 (0.24)
Property damage	-	-	-	1
Mean()	1,832	89.9	35.7	584
Standard-deviation()	2,489.95	77.30	31.24	286.57

*(.) Denote the p-value of the one-sided test.

() Mean and Standard-deviation are in millions of euros.

Correlation values are used as illustration of the dependence structure between the different LoB. As argued by Tang and Valdez (2005), the use of linear correlation in the insurance context remains very questionable. We note two negative correlations between losses from the motor-property damage and fire-third-party lines. It can be argued that the property damage line, providing cover for natural or human caused damages, and it is expected to be uncorrelated with motor. Regarding, the second negative correlation could be explained by the fact that the liability line belongs to the long term class whereas fire belongs to an intermediate class.The only significant correlation is between motor and Third-party lines.

Without loss of generality, we consider three liquid assets, namely two risky assets and a risk-free asset. The first one corresponds to the global CAC 40 index portfolio. The second asset, more risky, is approximated with a French stock fund. The risk-free asset is equivalent to the French Treasury Bills (3 months). Table 2 displays assets summary statistics over the same period of insurance claims data. We obtain the claims data from the French federation of insurance firms (FFSA) and the different assets quotes from DataStream.

Table 2. Summary statistics of annual return rate of assets

	Mean	St. dev	Pearson rho	Spearman rho	Kendall tau
T-Bills (3 months)	3.73%	-	-	-	-
CAC40 Global Index	6.82%	23%	0.23 (0.013)*	0.28 (0.005)	0.15 (0.007)
Stock fund	10.08%	59%			

*(.) Denote the p-value of the one-sided test.

The CAC40 Global index and stock fond are significantly and positively dependent. Table 3 presents various measures of dependence between aggregate insurance loss of the four Lob and investment risks. There is a weak negative dependence between the stock fund return rate and aggregate claims. Kendall tau is significantly different of zero at 5%.

Table 3. Dependence measures between aggregate claims and assets return rates

	CAC40 Global Index	Stock fund
Aggregate claims (Pearson Correlation)	-0.017 (0.10)*	-0.02 (0.15)
Aggregate claims (Spearman rho)	-0.004 (0.12)	-0.01 (0.085)
Aggregate claims (Kendall tau)	-0.031 (0.059)	-0.09 (0.037)

*(.) Denote the p-value of the one-sided test.

It is important to highlight the diversification effect between assets and liability risks coupled to this negative correlation.However, we should moderate the importance of this result since it depends heavily on the data, the sample size, period and line of business claims pattern. We approximate the rates of return of the two risky assets with a normal distribution and the aggregate claims follow with a gamma distribution[12]. We model the dependence between aggregate claims and return rate of stock fund with a Frank copula[13]. Thereafter, we construct the return rate of CAC40 by Cholesky decomposition. The simulated samples of aggregate claims, return rates of stock fund and CAC40 index are based on 10000 independent draws.

Since we do not consider all the lines of business, data related to the initial capital level is not exploitable. We chose an arbitrary initial capital value which verifies the following condition: when the whole of investable funds are invested in the minimum risk portfolio, the CVaR with a confidence level of 1% is nonpositive. Based on data, we choose an arbitrary value of 10000. The global technical provisions associated to the four lines of business are computed according to a value at risk criterion with a confidence level of 75% of the discounted aggregate claims distribution (see Lorent (2006) and Planchet et al. (2007)). To take into account the diversification effect between the LoBs, we calculate the VaR of aggregate loss rather than the sum of individual VaR. The premium loading factor and tax rate are equivalent to 20% and 35%, respectively. Regarding adjusting equity capital cost function, we use a proportional cost of 5%.

5. Results

We implement the methodology outlined above to find solutions. We proceed first to a graphical description of the optimal portfolio. We discuss next the optimum sensitivity to capital adjustment cost, initial capital size, aggregate claims volatility and dependence structure between liabilities and assets risks.

5.1 Efficient frontier

Figure 2 displays efficient frontiers (*ROC-CVaR*) and (*ROC-VaR*) without capital adjustement. All the curves are concave and strictly increasing. The (*ROC-CVaR*) curve is smoother than the (*ROC-VaR*) one,

[12] According to Hardy (2004): "There are two main types of aggregate claims approximation. The first matches the moments of the aggregate claims to the moments of a given distribution (for example, normal or gamma), and then use probabilities from the approximating distribution as estimates of probabilities for the underlying aggregate claims distribution.The second type of approximation assumes a given distribution for some transformation of the aggregate claims random variable".

[13] The use of copulas for modeling dependence structure between more that two random variables poses theoretical and technical problems. First, the complexity of deriving the parameter of the copula allowing for constructing random samples with the required dependence. Furthermore, the sub-identification issue raised since the unique parameter of the copula does not allow for modeling all possible dependence structures. To overcome these limits, a first part of dependence will be modeled with a bivariate copula and the second part will be described with the Pearson correlation given that the return rates of the two assets are normal.

especially for small risk values . As explained by Lemus et al. (1999), the non-convexity of value at risk causes computational difficulties and leads to erratic efficient frontier.

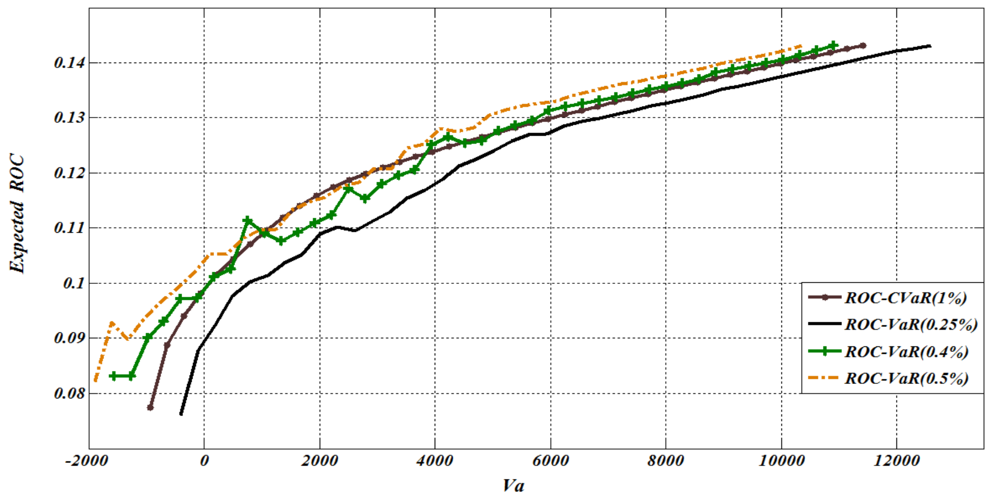

Fig. 2. ROC-CVaR and ROC-VaR efficient frontiers without capital adjustment and under different confidence levels.

Note, that for a given ROC, the CVaR(1%) efficient frontier is practically bounded by VaR(0.25%) and VaR(0.4%) ones. This result contradicts a common accepted view about the equivalence of CVaR(1%) and VaR(0.5%). Let us note that the choice for the initial capital has verified the condition that the CVaR of the minimum risk portfolio is negative. Figure 3 displays efficient portfolios verifying the zero-CVaR. These curves are obtained with the two step approach. After computing efficient portfolios from problem $[P_{\alpha}^{0}]$, we adjusted the ROC curve with the corresponding equity capital to obtain RORAC. As expected, the possible portfolios frontier for CVaR is quasiconcave and admits a unique optimum. Raising capital allows to insurer to improve the expected ROC from 9, 81% to 10, 13% when risk is measured with

CVaR.

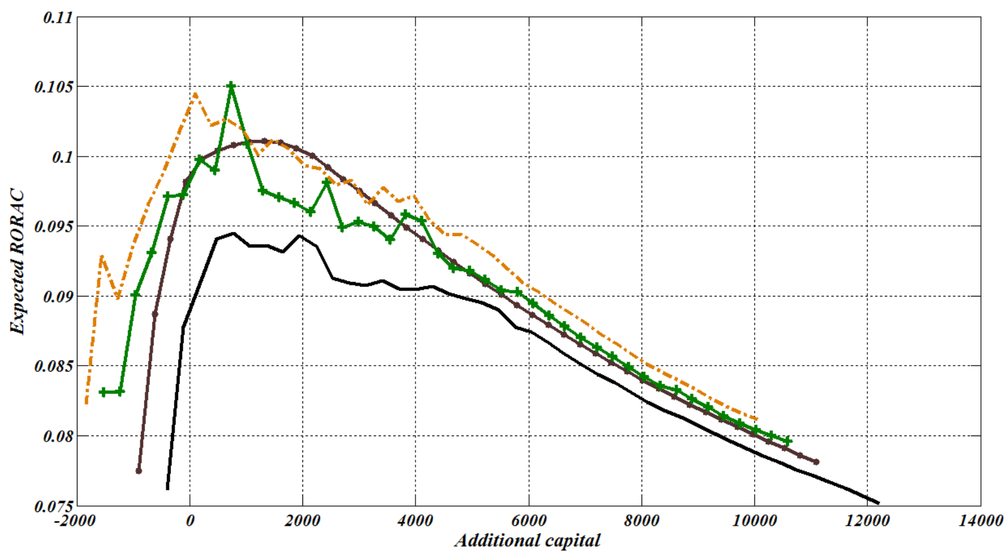

Fig. 3. Efficient portfolios curve verifiying a zero-CVaR and zero-VaR under different confidence levels.

These results are roughly similar to those of Planchet and Thereond (2007). Based on simulations, their objective criterion as a function of the one risky asset weight presents the same form as figure 3.

5.2 Optimum sensitivity analysis

We investigate now, optimum sensitivity to initial capital and other modelling assumptions such as claims variance, dependence structure between the riskiest asset return rate and the claims and extreme dependence.

5.2.1 Capital adjustment cost

To analyze the effect of capital adjusting cost on optimal solution, we choose different levels of adjustment capital cost. We note from figure 4 that with a proportional cost of 5% the maxima is at the right side of Y- axis. The more the cost increase the more the maximum gets close to Y-axis. When the cost becomes higher than a specific level (10.5% in our case), the maxima is in on the Y-axis.

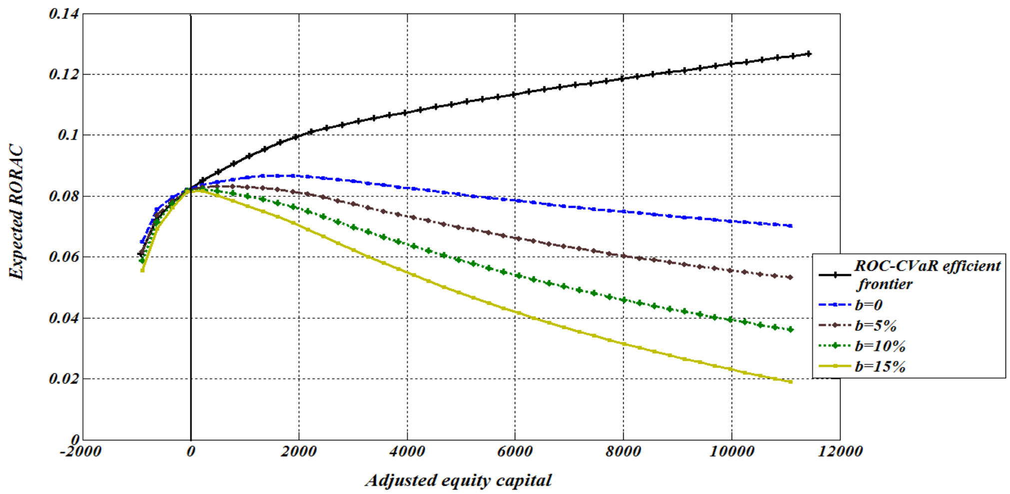

Fig. 4. ROC-CVaR efficient frontier without solvency constraint and efficient portfolios curve with different capital adjustment cost.

This finding suggests that the optimal decision will consist on keeping unchanged initial capital. Consequently, to carry out the solvency constraint, the insurer has to choose a particular portfolio that corresponds to the intersection between the efficient frontier without shortfall constraint and the Y-axis.

5.2.2 Initial capital level

Let us note first that risk is not affected by changes of initial capital levels. When initial capital is too high, the firm is over capitalized and RORAC is small. We observe from tables 4 and 5 that RORAC is a decreasing function of initial capital. We notice that for high level of capital there no capital adjustment.

Table 4. Optimum sensitivity to initial capital with a CVaR(1%)

C_0	ΔC_t^*	x_0^*	x_1^*	x_2^*	$RORAC$
10000	1335	0	0.14	0.86	10.11%
13000	0	0	0.77	0.23	9.97%
16000	0	0	0.66	0.34	9.38%
19000	0	0	0.60	0.4	8.90%
21000	0	0	0.58	0.42	8.64%

The insurer has enough guaranties to attain any optimal risky portfolio. There is a transfer of funds from the CAC 40 to stock fund.

Table 5. Optimum sensitivity to initial capital with a VaR(0.5%)

C_0	ΔC_t^*	x_0^*	x_1^*	x_2^*	$RORAC$
10000	699	0	0.23	0.77	10.89%
13000	0	0	0.33	0.67	10.33%
16000	0	0	0.42	0.58	9.63%
19000	68	0	0.45	0.55	9.05%
21000	127	0	0.48	0.52	8.76%

In comparing Tables 4 and 5, we see some differences between optimal solutions depending on the risk metric. This disparity tends to decrease when initial capital is high.

5.2.3 aggregate Claims variance

The lower (upper) volatility corresponds to the aggregate claims standard-deviation with perfect negative (positive) correlation matrix between the four LoBs. Tables 6 and 7 display the optimal solutions sensitivity to aggregate claims volatility. We note first, that when volatility increases the expected decrease and optimal capital adjustment increases.

Table 6. Optimum sensitivity to aggregate claims volatility
with a CVaR(1%)

σ_S	ΔC_t^*	x_0^*	x_1^*	x_2^*	$RORAC$
2062	0	0.03	0.12	0.85	11%
2268	715	0.1	0.15	0.74	10.32%
2473	2214	0	0.2	0.8	9.69%
2679	3415	0	0.24	0.76	9.12%
2885	4747	0	0.29	0.71	8.6%

The volatility raise implies a more important risk and hence the need of more additional equity capital. The optimal allocation in response to claim volatility variation is more regular than in the case of initial capital variation. Let us note a fight to the risky assets.

Table 7. Optimum sensitivity to aggregate claims volatility
with a VaR(0.5%)

σ_S	ΔC_t^*	x_0^*	x_1^*	x_2^*	$RORAC$
2062	7	0	0.25	0.75	11.56%
2268	31	0.15	0.27	0.58	11.16%
2473	1284	0.04	0.29	0.67	10.52%
2679	2283	0	0.275	0.725	9.89%
2885	3532	0	0.34	0.65	9.34%

5.2.4 Dependence structure

Dependence structure variation between aggregate claims and stock fund return rates affects the firm global risk. In the case of co-monotonicty between the two variables and because of their inverse relationship the risks are offseted. There is no additional capital which reflects the large firm capacity to make risky investment and improve its RORAC. The opposite case involves a positive adjusting capital.

Table 8. Optimum sensitivity to aggregate claims and stocks fund return rates dependence structure with a CVaR(1%)

Dependence	ΔC_t^*	x_0^*	x_1^*	x_2^*	$RORAC$
Counter-monotonic	2016.8	0.37	0	0.63	8.22%
Frank $(\alpha \prec 0)$	1404.8	0.14	0	0.86	9.23%
Independent	1064.3	0.04	0.19	0.77	10.38%
Frank $(\alpha \succ 0)$	0	0	0.46	0.54	12.69%
Co-monotonic	0	0	0.6	0.4	13.2%

Table 9. Optimum sensitivity to aggregate claims and stocks fund return rates dependence structure with a VaR(0.5%)

Dependence	ΔC_t^*	x_0^*	x_1^*	x_2^*	$RORAC$
Counter-monotonic	1532.3	0.28	0	0.72	8.73%
Frank $(\alpha \prec 0)$	1061.8	0	0	1	9.8%
Independent	310.8	0	0.22	0.78	11.21%
Frank $(\alpha \succ 0)$	0	0	0.55	0.45	13.07%
Co-monotonic	0	0	0.63	0.37	13.33%

6. Conclusion

This paper proposes two strong theoretical approaches to solve a jointly portfolio selection and capital requirement problem for non life insurance firms. The optimality is defined in terms of maximization of the expected return on risk-adjusted capital (RORAC) subject to zero-conditional value at risk shortfall constraint. We derive a closed form relationship between economic capital and optimal portfolio. It turns out that active equity capital management allows insurers to attain higher expected RORAC compared with that with fixed capital. The optimal portfolio verifying shortfall constraint is function of surplus, distributions and dependence structure between risks and capital raising capacity. Furthermore, we show that the numerical results are robust to the relaxation of the assumption regarding equity adjustment allocation.

This study presents a limitation regarding the tightness of the sample size. Our analysis could be generalized to study investment opportunities such as M&A, lunching new business and to any project requiring new equities either to finance it or to reduce to aggregate firm risk. Because of time variation of risks distributions and dependence structure, It might also be desirable in future work to analyze and solve the problem in a multi-period framework where others assets and more detailed taxation could be considered.

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