Sensitivity Analysis of Credit Risk Measures in the Beta Binomial Framework

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Abstract

This paper reconsiders the Beta Binomial approach for modeling default risk in homogenous credit portfolio. The beta mixing distribution is viewed as a function of the common default probability and the common default correlation. We mainly focus on the correlation parameter and provide closed-form expressions for sensitivities of key credit risk indicators. Sensitivity and elasticity analysis then show that the common default correlation impacts on the credit at risk and expected shortfall quite differently. A final application is performed on CDOs.

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SENSITIVITY ANALYSIS OF CREDIT RISK MEASURES IN THE BETA BINOMIAL FRAMEWORK

Abstract: This paper reconsiders the Beta Binomial approach for modeling default risk in homogenous credit portfolio. The beta mixing distribution is viewed as a function of the common default probability and the common default correlation. We mainly focus on the correlation parameter and provide closed-form expressions for sensitivities of key credit risk indicators. Sensitivity and elasticity analysis then show that the common default correlation impacts on the credit at risk and expected shortfall quite differently. A final application is performed on CDOs.

JEL Codes: G10, G11.

1 Introduction

Mixed Binomial Models, also known as Bernoulli Mixture Models, are common ways to model default risk in credit portfolios. To account for default dependency, the default probability is assumed randomly distributed according to a so-called mixing distribution. These frameworks are usually considered as very appealing by modelers because they are easy to simulate in Monte Carlo analysis and simple to calibrate on real data (Frey and McNeil (2003)). And, indeed, many standard industry models for managing credit portfolios are nothing else than specific Bernoulli mixture models[1]. Among these, the Beta Binomial Approach is of specific nature for credit portfolio managers because it serves as benchmark in horserace of other models.

This paper reconsiders the Beta Binomial Approach and exploits in depth a reparametrized version of the Beta Mixing Distribution[2]. Its aim is to develop a financial and quantitative analysis with no reference to statistical shape parameters of the mixing distribution[3]. Both the expected default probability and the default correlation are favored as key input parameters. In what follows, one insists on the common default correlation between issuers for several reasons. First of all, the expected default probability is often considered as fixed in homogenous credit portfolios. Second, the default correlation may vary for a given level of default risk (see Renault and Servigny (2004) for documented statistics). Third, in view of the literature, it is not clear how sensitive are classical models with respect to that variable.

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[1] A mixing Logit-Normal distribution is explicitly used in *CreditPortfolioView* (see Wilson (1997a, b)) while it is a Probit-Normal one in *Creditmetrics*. In addition, Frey and McNeil (2002) have demonstrated that the CreditRisk+ solution implicitly uses a beta mixing distribution for the default probability.

[2] It therefore admits some connections with CreditRisk+ but this point is left for future research.

[3] This feature is desirable because everybody involved in the credit industry is not necessary "fluent" in statistics. Moreover, it is not so evident that people involved in the credit business interpret Beta shape parameters in the same way. It is well known (see, e.g., Frey and McNeil (2001)) that fixing any two of the first moment of the random default probability, its second moment or the default correlation determines shape parameters of the beta mixing distribution. However, no research has developped further this way of reasoning.

Armed with this approach, easy-to-implement analytical expressions are obtained for analyzing the sensitivity of standard credit indicators to the default correlation. Sensitive analysis without recourse to simulation is highly welcome by credit analysts for several reasons. First of all, they can speed up computation and subsequent decisions making. Second, it is well known that simulation-based approaches admit drawbacks. Essentially, results depend on the number of simulation runs and the way the random figures are generated so that they can be fairly unstable and even fail to give definitive conclusions on the tails' behavior. The present research admits closed connections with a current stream of studies dedicated to the non constant volatility (see Burtschell, Gregory and Laurent (2007) for references). It can be viewed as a simple approach to assess the impact of a correlation shift.

Following standard practices, we mainly focus on Credit at Risk and the Expected Shortfall (see Szego (2005). Sensitivities and elasticities are studied with respect to the sole common default correlation rather than the two statistical shape parameters of the distribution. Quantitative analysis show that the correlation coefficient parameter plays an essential role but also that it impacts quite differently on the considered credit risk measures. Sensitivities of the credit at risk and the tail function appear either positive or negative while that of the expected shortfall remains positive. To highlight further this key role of the common default correlation, we examine the asymptotic tail function associated to different tranches of CDOs. And we show how different holders of the tranche are impacted by this parameter.

The rest of the paper proceeds as follows. Section 2 presents the standard framework for analyzing homogenous credit portfolio. Section 3 introduces the reparametrized beta mixing distribution we suggest and analyzes homogenous credit portfolios. Section 4 considers large portfolios and reviews standard credit risk indicators. Section 5 then undertakes sensitivity analysis of these credit risk measures to the default correlation.

2 The standard framework

We consider in what follows a homogenous credit portfolio of N loans or bonds. In this paper, homogeneity refers essentially to both the credit profile of borrowers and the design of credits... It is assumed that credit ratings are known and identical within the credit portfolio. The same is true for recovery rates or equivalently losses given default. By denoting by T the investment period and τ_i the default time of the $i-$th borrower, the variable $X_i = 1_{\tau_i < T}$ plays the role of a default indicator. If face values are equal to 1, the value loss (suffered at the end of the investment period by the holder of the credit portfolio) is equal to the number of defaults. So it can be described by the sum of indicators $N_{def}(N) = \sum_{i=1}^{N} X_i$. Note that this latter assumption prevents tricky notations without modifying the salient feature of the credit risk modelling.

2.1 Mixed binomial models for credit risk portfolios

Every loans or bonds have the same rating meaning that they share the same probability of default p. As a result, the above default indicators are identical Bernoulli distributed variables. And the number of default is a binomial variable with parameters $(N,\ p)$. More precisely, this random variable denoted by $N_{def}(N)$ simply takes values between 0 (no default) and N (all firms default) with a probability density described by:

$$\Pr\left[N_{def}(N) = j\right] = \binom{N}{j} p^j (1-p)^{N-j}, \qquad j \in \{0, 1, ..., N\} \tag{1}$$

$\binom{N}{j} = C_j^N = \dfrac{N!}{j!\,(N-j)!}$ stands for the number of pairs of j defaults among the N borrowers. The cumulative density function is then straightforwardly given by $\Pr\left[N_{def}(N) \leq k\right] = \sum_{j=0}^{k} \Pr\left[N_{def}(N) = j\right]$. The mean loss is $E\left[N_{def}(N)\right] = N \times p$, and its variance $\sigma^2\left[N_{def}(N)\right] = N \times p \times (1-p)$. Hence, the average number of default is proportional to the number of borrowers and its standard deviation to the square root of N.

The mixed binomial framework introduces dependence among default by letting the common default probability to be stochastic. If one assumes conditional independence of individual defaults (given the probability of default), then the probability of facing k

defaults is then given by

$$\Pr\left[N_{def}\left(N\right) \leq k\right] = E\left[\Pr\left[N_{def}\left(N\right) \leq k \middle| p\right]\right] = \int_0^1 \sum_{j=0}^{k} \binom{N}{j} p^j \left(1-p\right)^{N-j} f\left(p\right) dp.$$

where f is the mixing distribution. Such a distribution is clearly central to model the default probability and the resulting dependance between defaults. As recalled below, it is also a good proxy for the (percentage) loss distribution of large homogenous credit portfolio. The introduction recalls that the Beta mixing distribution is a classical way to randomize the default probability p. For readers' convenience, one finds useful to present few results, before introducing our own parametrization.

2.2 The standard Beta Binomial Approach

The standard Beta mixing distribution assumes that the probability density function of the default probability is well described by:

$$f\left(p;\alpha,\beta\right) := \frac{p^{\alpha-1}\left(1-p\right)^{\beta-1}}{\int_0^1 p^{\alpha-1}\left(1-p\right)^{\beta-1} dp} = \frac{\Gamma\left(\alpha+\beta\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} p^{\alpha-1}\left(1-p\right)^{\beta-1}$$

where shape parameters α, β are positive real numbers and Γ is the Gamma function. Properties of the Beta distribution are well known. Its probability density function is humped, skewed and leptokurtic. The α shape parameter controls the steepness of the hump while the β parameter controls the fatness of the tail. The expected default probability and associated variance are respectively given by $E\left[\mathbf{p}\right] := \int_0^1 pf\left(p;\alpha,\beta\right)dp = \frac{\alpha}{\alpha+\beta}$ and $V\left[\mathbf{p}\right] = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$. The skewness is $Sk\left[\mathbf{p}\right] = \frac{2(\beta-\alpha)}{\sqrt{\rho}\sqrt{\alpha\beta}(\alpha+\beta+2)}$. The corresponding cumulative density function is known as the regularized incomplete beta function:

$$I_x\left(\alpha,\beta\right) := \frac{\int_0^x p^{\alpha-1}\left(1-p\right)^{\beta-1} dp}{\int_0^1 p^{\alpha-1}\left(1-p\right)^{\beta-1} dp} := \frac{B_x\left(\alpha,\beta\right)}{B\left(\alpha,\beta\right)}$$

where $B\left(\alpha,\beta\right)$ and $B_x\left(\alpha,\beta\right)$ stand for the so-called beta function and incomplete beta function respectively. Many useful identities and recurrence results exist on these functions and we refer to Abramowitz and Stegun (1972) for details. The incomplete beta distribution admits useful relations with the generalized hypergeometric function since $B_x\left(\alpha,\beta\right) = \frac{1}{\alpha}x^{\alpha}{}_2F_1\left(\alpha,1-\beta,\alpha+1;x\right)$.

From a credit management viewpoint, the dependence between default events is the second dimension of interest in a homogenous credit portfolio - the first one being the expected default probability. It is useful to emphasize the following known result.

Proposition 1 *In the mixed beta binomial framework, the common default correlation of an homogenous credit portfolio is*

$$cor\left[X_i, X_j\right] = \frac{1}{1+\alpha+\beta} := \rho.$$

Since α and β are strictly positive, $\rho > 0$.

Proof. There are different ways to demonstrate this result. The following proof is straightforward. It is well known that:

$$cov\left[X_i, X_j\right] = cov\left[E\left[X_i \middle| \mathbf{p}\right], E\left[X_j \middle| \mathbf{p}\right]\right] + E\left[\mathrm{cov}\left[X_i, X_j \middle| \mathbf{p}\right]\right]$$

Because of the conditional independence, the second term is null. $cov\left[E\left[X_i \middle| \mathbf{p}\right], E\left[X_j \middle| \mathbf{p}\right]\right]$ is equal to $\mathrm{cov}[\mathbf{p}, \mathbf{p}] = V\left[\mathbf{p}\right]$ for $i \neq j$. Correlation definition then yields to:

$$\mathrm{cor}\left[X_i, X_j\right] = \frac{V\left[\mathbf{p}\right]}{E\left[\mathbf{p}\right]\left(1 - E\left[\mathbf{p}\right]\right)} = \frac{E\left[\mathbf{p}^2\right] - E\left[\mathbf{p}\right]^2}{E\left[\mathbf{p}\right] - E\left[\mathbf{p}\right]^2}, \qquad i \neq j. \tag{2}$$

■

Clearly, the availability of analytical results makes the beta binomial framework suitable for modeling homogenous credit portfolios. For known shape parameters (α, β), properties of the beta distribution are well known and $E\left[\mathbf{p}\right]$, $V\left[\mathbf{p}\right]$ and ρ are easy to compute with previous expressions. For instance, if $\alpha = \beta = \gamma$ then the beta distribution is symmetric with respect to $E\left[\mathbf{p}\right] = \frac{1}{2}$. Straightforward computations give $V\left[\mathbf{p}\right] = \frac{1}{4(2\gamma+1)}$ and $\mathrm{cor}[X_i, X_j] = \frac{1}{1+2\gamma} = 4V\left[\mathbf{p}\right]$, meaning that the standard deviation of the (random) default probability is bounded by $\frac{1}{2}$. This limit case corresponds to $\gamma \approx 0$ for which $\mathrm{cor}[X_i, X_j] = 1$. In such a case, either all issuers survive, either all defaults. The corresponding distribution weights only 0 and 1. If instead, $\gamma = 1$, the symmetric beta distribution is the uniform one with $V\left[\mathbf{p}\right] = \frac{1}{12}$ and $\mathrm{cor}[X_i, X_j] = \frac{1}{3}$.

Nevertheless, dealing with two shape parameters is not so comfortable from a management viewpoint. Beyond the possible lack of understanding, key indicators for credit portfolio may behave differently as we change α and β. Typically, sensitivities of the mean default probability $\frac{\partial E[\mathbf{p}]}{\partial \alpha} = \frac{\beta}{(\alpha+\beta)^2}$ and $\frac{\partial E[\mathbf{p}]}{\partial \beta} = -\frac{\alpha}{(\alpha+\beta)^2}$ are respectively positive and negative.

Because the default probability is essentially fixed in an homogenous portfolio, we can rewrite $\beta(\alpha) = \alpha \frac{1-E[\mathbf{p}]}{E[\mathbf{p}]}$ to limit such a complexity. And, in that case, $V[\mathbf{p}] = \frac{E[\mathbf{p}]^2(1-E[\mathbf{p}])}{\alpha+E[\mathbf{p}]}$ and $\mathrm{cor}[X_i, X_j] = \frac{1}{1+\frac{\alpha}{E[\mathbf{p}]}}$. Hence (given $E[\mathbf{p}]$) both the variance of $\mathbf{p}$ and the default correlation are decreasing functions of α. This paper rather suggests an alternative approach that makes the beta distribution a function of the common default correlation between issuers. As far as we know, such a parametrization has not been exploited anywhere else. This is the aim of the paper.

3 A correlation based Beta mixing distribution for homogenous credit portfolios

Mixed Beta binomial models may be viewed as functions of the common default probability and the common default correlation ρ. To see this, it is sufficient to note that results of the previous section yields to:

$$\begin{cases} \alpha = E[\mathbf{p}] \frac{1-\rho}{\rho} \\ \beta = (1-E[\mathbf{p}]) \frac{1-\rho}{\rho} \end{cases} \tag{3}$$

The Beta distribution can then be reparameterized as $f(p;\alpha,\beta) = \varphi(p; E[\mathbf{p}], \rho)$. What follows goes a step further because the mean probability of default $E[\mathbf{p}]$ is essentially constant in homogenous credit portfolios. For an homogenous credit portfolio, one therefore considers:

$$\psi_{E[\mathbf{p}]}(p;\rho) \equiv f\left(p; E[\mathbf{p}] \frac{1-\rho}{\rho}, (1-E[\mathbf{p}]) \frac{1-\rho}{\rho}\right).$$

Such a parametrization allows one to rephrase in financial terms most of well known properties. For instance, by virtue of proposition 1, the variance of the default probability is now a simple increasing function of the default correlation given by $V[\mathbf{p}] = \rho E[\mathbf{p}](1 - E[\mathbf{p}])$. So the variance first increases with $E[\mathbf{p}]$ from 0 to $\frac{\rho}{4}$ (obtained for $E[\mathbf{p}] = \frac{1}{2}$) and then decreases to zero as $E[\mathbf{p}]$ gets to one[4]. The skewness, rewritten as $Sk = 4\frac{\frac{1}{2}-E[\mathbf{p}]}{\sqrt{E[\mathbf{p}](1-E[\mathbf{p}])}}\frac{\sqrt{\rho}}{1+\rho}$, highlights in very simple way, that the distribution is symmetric (i.e. with a null skewness) for $E[\mathbf{p}] = \frac{1}{2}$. All expressions of section 2 can also be rewritten in terms of the sole correlation parameter. The probability that j credit(s) default in the portfolio becomes

$$\Pr[N_{def}(N) = j] = \int_0^1 \binom{N}{j} p^j (1-p)^{N-j} \psi_{E[\mathbf{p}]}(p;\rho)\, dp \tag{4}$$

To illustrate in a first and simple way the key role of the common default correlation, Figure 1 compares graphically the mixed beta binomial distribution parametrized by the correlation coefficient with the binomial density given in equation 1. Figure 1 considers an homogenous credit portfolio of $N = 100$ loans or bonds. The shadow probability density function corresponds to the straight binomial model (i.e. the non dependent case), while other ones correspond to the reparameterized beta binomial model. We set $E[p]$ to 10% and the correlation parameter ρ to either 2.5% or 10% (left and right graphs respectively). Clearly, the default correlation impacts significantly distributions. The probability density functions associated to the dependent case are clearly skewed; their (right) tails are heavier than the ones of the independent case.

Insert Figure 1 about here.

For completeness, Table 1 provides the probability of k defaults within a portfolio of 10 assets ($\Pr[N_{def}(N) = k]$ with $N = 10$) for different values of default correlation given that the expected default probability is equal to 5% in all cases. All figures are expressed in %. The common default correlation ranges from about 0 to 10%. Such values are admissible in view of Table 5.2 of Renault and Servigny (2004). These authors report,

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[4] Note however that, except speculators, very few investors would invest in credit portfolios with too significant expected default probability.

on the basis of "Standard & Poor's CreditPro" data, that the (1-year) default correlation within a given rating class is higher than 0% (AAA) and lower than 8.97% (CCC). These figures are estimated on observed defaults between 1981 and 2002. An interesting feature merits to be emphasized as the default correlation increases. Both the probability of no default in portfolio and the probability of the larger number of defaults increases. This is easily explained by the fact that, as the default correlation rises, underlying bonds or loans behave more and more similarly. Interestingly, one can observe that, for $k = 2$, the probability of k defaults viewed as a function of ρ is first decreasing and then decreasing. This point is explored in the last section.

Insert Table 1 about here.

We can also add results on the total number of defaults $(N_{def}(N))$ in the homogenous credit risk portfolio or equivalently on the loss rate in the portfolio $L(N)$. $L(N)$ is the proportion of default $\frac{N_{def}(N)}{N}$.

Proposition 2 *The total number of default among the N issuers verifies:*

$$E\left[N_{def}(N)\right] = NE\left[\mathbf{p}\right] \tag{5}$$

$$V\left[N_{def}(N)\right] = \left[N + N(N-1)\rho\right] E\left[\mathbf{p}\right]\left(1 - E\left[\mathbf{p}\right]\right) \tag{6}$$

The mean and the variance of the loss rate are respectively:

$$\begin{aligned} E\left[L(N)\right] &= E\left[\mathbf{p}\right] \\ V\left[L(N)\right] &= \frac{N + N(N-1)\rho}{N^2} E\left[\mathbf{p}\right]\left(1 - E\left[\mathbf{p}\right]\right) \end{aligned}$$

The variance of the loss rate $L(N)$ is decreasing with the number of credits and tends to $\rho E\left[\mathbf{p}\right]\left(1 - E\left[\mathbf{p}\right]\right) \equiv V\left[\mathbf{p}\right]$.

Proof.

$$\begin{aligned}
V\left[N_{def}\left(N\right)\right] &= V\left[E\left[\left.N_{def}\left(N\right)\right|\mathbf{p}\right]\right] + E\left[V\left[\left.N_{def}\left(N\right)\right|\mathbf{p}\right]\right] \\
&= N^2 V\left[\mathbf{p}\right] + E\left[NV\left[\left.X_1\right|\mathbf{p}\right]\right] \\
&= N^2 V\left[\mathbf{p}\right] + NE\left[\mathbf{p}\left(1-\mathbf{p}\right)\right] \\
&= N^2 V\left[\mathbf{p}\right] + NE\left[\mathbf{p}\right] - NE\left[\mathbf{p}^2\right]
\end{aligned}$$

The second moment being $E\left[\mathbf{p}^2\right] = \rho E\left[\mathbf{p}\right] + \left(1-\rho\right) E\left[\mathbf{p}\right]^2$. Results on $L\left(N\right)$ are straightforward consequences. ■

The above results imply that the variance of the loss rate $V\left[L\left(N\right)\right]$ is a strictly increasing function of ρ with a minimum and a maximum given by $\frac{1}{N}E\left[\mathbf{p}\right]\left(1-E\left[\mathbf{p}\right]\right)$ and $E\left[\mathbf{p}\right]\left(1-E\left[\mathbf{p}\right]\right)$ respectively. The variance of the loss rate is a decreasing function of the number of credits in the portfolio with a minimum and a maximum given by $\frac{1+\rho}{2}E\left[\mathbf{p}\right]\left(1-E\left[\mathbf{p}\right]\right)$ and $\rho E\left[\mathbf{p}\right]\left(1-E\left[\mathbf{p}\right]\right)$ respectively. Because of this feature, the loss rate variable appears particularly suitable for analyzing credit risk portfolios (compared to its alternative that is the number of defaults $N_{def}\left(N\right)$). To elaborate on these properties, Figure 2 puts together a couple of graphs. The left one draws the normalized volatility of the loss rate as a function of the common default correlation for different numbers of credits in the portfolio. The normalized volatility is computed by $\sqrt{\frac{V[L(N)]}{E[\mathbf{p}](1-E[\mathbf{p}])}}$. This graph displays how the normalized volatility tends to the minimum normalized volatility given by $\sqrt{\rho}$. The right graph is inspired from the traditional portfolio theory. It plots the normalized variance of the loss rate as a function of the number of credits. This graph displays the diversification effect within a credit portfolio. Like more standard (stocks) portfolios, the variance decreases and tends to a non-zero value.

Insert Figure 2 about here.

4 Analyzing large homogenous portfolios

Schönbucher (2003) have well explained that, as the number of assets in the credit portfolio becomes large, the loss rate statistic becomes the relevant figure to consider. The proportion of defaults (whose conditional expectation is p for every N) tends to p as N gets large[5]. Models for large credit risk portfolios thus routinely exploit the fact that, for large N, $\Pr\left[L\left(N\right)\leq l\right]\approx\Pr\left[p\leq l\right]$. In other words, tails of the true loss (rate) distribution of large homogeneous credit portfolios may be approximated by the tail of the mixture distribution. Our setting allows one to reconsider the loss rate distribution in the light of the common default correlation. The loss rate distribution in large homogenous credit portfolios is described by:

$$\Pr\left[L\leq l\right]=\int_0^l\psi_{E[\mathbf{p}]}\left(p;\rho\right)dp:=\mathcal{I}_{E[\mathbf{p}]}\left(l;\rho\right).$$

The tail function, defined by $\mathrm{TF}_{E[\mathbf{p}]}\left(l;\rho\right)=\Pr\left[L>l\right]=1-\mathcal{I}_{E[\mathbf{p}]}\left(l;\rho\right)$ can provide a first approach to understand the extreme risk in credit portfolios. Standard credit risk measures for analyzing credit portfolios are also related to the above probability. Denoting by c a confidence level (typically 99%, 99.9%), the Credit at Risk $CaR_c\left(L\right)$ is the value such that $\mathrm{TF}_{E[\mathbf{p}]}\left(CaR_c\left(L\right);\rho\right)=\Pr\left[L>CaR_c\left(L\right)\right]=1-c$ or equivalently

$$\Pr\left[L\leq CaR_c\left(L\right)\right]=\mathcal{I}_{E[\mathbf{p}]}\left(CaR_c\left(L\right);\rho\right)=c \tag{7}$$

This is the $c-th$ quantile of the reparametrized Beta distribution. The cumulative density function $l\mapsto\mathcal{I}_{E[\mathbf{p}]}\left(l;\rho\right)$ being continuous, this may be rewritten $CaR_c\left(L\right)=\mathcal{I}_{E[\mathbf{p}]}^{-1}\left(c;\rho\right)$. We can remind readers that the Credit at risk is closely related to the regulatory capital issue. The expected shortfall is another important indicator to consider. Defined by $ES_c=E\left[L\middle|L\geq CaR_c\left(L\right)\right]$ it has more desirable properties than the $CaR_c\left(L\right)$, as explained by Artzner et al. (1999) and discussed by Szego (2005). This coherent measure of risk can be computed in the present framework by a couple of ways:

$$\begin{aligned} ES_c\left(\rho\right) &= \frac{1}{1-c}\int_c^1 CaR_u\left(L\right)du=\frac{1}{1-c}\int_c^1\mathcal{I}_{E[\mathbf{p}]}^{-1}\left(u;\rho\right)du && (8)\\ ES_c\left(\rho\right) &= CaR_c\left(L\right)+\frac{1}{1-c}\int_{CaR_c(L)}^1\left[1-\mathcal{I}_{E[\mathbf{p}]}\left(l;\rho\right)\right]dl && (9)\end{aligned}$$

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[5] To see this, it is sufficient to note that the conditional variance of the proportion $\sigma^2\left[\frac{N_{def}(N)}{N}\middle|p\right]=\frac{p(1-p)}{N}$ tends to 0.

One can remark that both requires numerical integration techniques[6].

Figure 3 puts together four different graphs that display the loss rate probability distribution function (upper left), the tail function (upper right), the Credit at Risk (lower left) and the Expected Shortfall (lower right) of a large homogenous credit portfolio with identical average default probability. Following usual practices, the tail function is provided on a logarithm $y-$scale. In the same graph, the horizontal line stands for the 99-th percentile and serves as a benchmark. One can verify that the expected shortfall is larger than the Credit at Risk as predicted by equation (9). All these graphs illustrate that the default correlation impacts significantly on the perceived risk of the large homogeneous portfolio (as measured by the different credit indicators).

Insert Figure 3 about here.

Beyond this graphical approach, it is worth to assess quantitatively the sensitivity of credit indicators with respect to the default correlation. This is the aim of the following section. Before introducing analytical expressions, Table 2 provides direct percentage differences of credit at risk and expected shortfall for a reference default correlation of 1.25%. As becomes clear in this table, the credit risk assessment is dramatically impacted by any misestimation in default correlation. E.g. in a credit portfolio with a 5% expected default probability, the credit at risk for a 10% default correlation is 2 times larger than that computed for a 1.25% correlation. The expected shortfall is even about 2.5 times larger. As suggested by Figure 3, percentage errors are worse as the confidence level c increases. We can notice however that errors for the CaR and the ES become of same order for huge confidence level. Interestingly, the largest correlation case displays a maximum percentage errors for the expected shortfall at the $c = 99.99\%$ level. Further experimentation reveals that the same is true for the Credit at Risk but at a (even) larger confidence level.

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[6] The latter appears less time-consumming than the former because no inversion is required. This remark may be helpful for whom wants to draw intensive graphics.

Insert Table 2 about here.

5 Sensitivity Analysis of credit risk indicators

This section provides closed-form formulae for credit indicators to analyze their sensitivity to default correlation. Our reparametrization of the Beta distribution suggests to write the cumulative density function and its inverse function as $\mathcal{I}_{E[\mathbf{p}]}(l;\rho) = I_x(\alpha,\beta)$ and $\mathcal{I}_{E[\mathbf{p}]}^{-1}(c;\rho) = I_x^{-1}(\alpha,\beta)$ respectively. Due to expressions (7), (8) and (9), derivatives formulae are available to the extent we can compute $\frac{\partial I_x(\alpha,\beta)}{\partial\alpha}$, $\frac{\partial I_x(\alpha,\beta)}{\partial\beta}$, $\frac{\partial I_x^{-1}(\alpha,\beta)}{\partial\alpha}$ and $\frac{\partial I_x^{-1}(\alpha,\beta)}{\partial\beta}$. Some expressions are exposed in the appendix. One then finds:

$$\begin{aligned}\frac{\partial\mathcal{I}_{E[\mathbf{p}]}(l;\rho)}{\partial\rho} &= \frac{\partial I_x}{\partial\alpha}\frac{\partial\alpha}{\partial\rho} + \frac{\partial I_x}{\partial\beta}\frac{\partial\beta}{\partial\rho} \\ &= -\frac{1}{\rho^2}\left[E[\mathbf{p}]\frac{\partial I_x}{\partial\alpha}\left(E[\mathbf{p}]\frac{1-\rho}{\rho},(1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right. \\ &\quad \left.+(1-E[\mathbf{p}])\frac{\partial I_x}{\partial\beta}\left(E[\mathbf{p}]\frac{1-\rho}{\rho},(1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right]\end{aligned}$$

Other analytical expressions are derived along similar lines. The sensitivity of the tail function with respect to default correlation is simply given by $\frac{\partial \mathrm{TF}_{E[\mathbf{p}]}(l;\rho)}{\partial\rho} = -\frac{\partial\mathcal{I}_{E[\mathbf{p}]}(l;\rho)}{\partial\rho}$. The $CaR_c(\rho)$ sensitivity to the default correlation is assessed by:

$$\begin{aligned}\frac{\partial CaR_c(\rho)}{\partial\rho} &= \frac{\partial\mathcal{I}_{E[\mathbf{p}]}^{-1}(c;\rho)}{\partial\rho} \\ &= -\frac{1}{\rho^2}\left[E[\mathbf{p}]\frac{\partial I_c^{-1}}{\partial\alpha}\left(E[\mathbf{p}]\frac{1-\rho}{\rho},(1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right. \\ &\quad \left.+(1-E[\mathbf{p}])\frac{\partial I_c^{-1}}{\partial\beta}\left(E[\mathbf{p}]\frac{1-\rho}{\rho},(1-E[\mathbf{p}])\frac{1-\rho}{\rho}\right)\right]\end{aligned}$$

The sensitivity of the expected shortfall admits a couple of expression depending on the considered definition. One finds either

$$\frac{\partial ES_c(\rho)}{\partial\rho} = \frac{1}{1-c}\int_c^1 \frac{\partial\mathcal{I}_{E[\mathbf{p}]}^{-1}(u;\rho)}{\partial\rho}du$$

or

$$\frac{\partial ES_c(\rho)}{\partial \rho} = -\frac{1}{1-c}\int_{VaR_c(L)}^{1} \frac{\partial \mathcal{I}_{E[\mathbf{p}]}(l;\rho)}{\partial \rho} dl$$

This latter (perhaps surprisingly simple) expression comes from the differentiation of equation (9) with respect to ρ and simplification

$$\begin{aligned}\frac{\partial ES_c(\rho)}{\partial \rho} &= \frac{\partial CaR_c(\rho)}{\partial \rho} - \frac{1}{1-c}\int_{CaR_c(\rho)}^{\infty} \frac{\partial \mathcal{I}_{E[\mathbf{p}]}(l;\rho)}{\partial \rho} dx \\ &\quad -\frac{1}{1-c}\frac{\partial CaR_c(\rho)}{\partial \rho}\left[1-\mathcal{I}_{E[\mathbf{p}]}(CaR_c(\rho);\rho)\right] dx\end{aligned}$$

Once again, it is expected to be less time-consuming because no inversion is involved. However, the following analysis favors the former expression because it involves the same underlying quantile function as the Credit at Risk. Note that, every formulae have been checked with approximate numerical derivatives. Armed with these expressions, one can comfortably undertake a sensitivity analysis of the Credit at Risk and Expected Shortfall in Figure 4.

Insert Figure 4 about here.

Figure 4 displays interesting results concerning the credit measures sensitivities to the default correlation. The four graphs suggest that the two measures are impacted differently. First of all, the derivative of the Credit at Risk is first negative and then becomes positive while it is always positive for the expected shortfall. Secondly, the higher default correlation case appears to the more sensitive to the default correlation when considering the absolute value of the first derivative. Things appear a bit different for elasticities for which the ordering can substantially changed. E.g. the $ES_c(L)$ is less elastic to the common default correlation.

6 Applications to CDOs

Portfolios with a limited number of credits are obviously common in asset management. Typically, credit portfolios underlying CDOs contains 125 different names. The above analysis can be easily extended to account for this. Technically, the exercise is straightforward and, for instance, the probability that no more than k credits default within the portfolio of size N is given by:

$$\Pr\left[N_{def}\left(N\right) \leq k\right] = \sum_{j=0}^{k} \int_{0}^{1} \binom{N}{j} p^{j} \left(1-p\right)^{N-j} \psi_{E[\mathbf{p}]}\left(p;\rho\right) dp$$

The expression clearly highlights that the dependence to the common default correlation comes from $\psi_{E[\mathbf{p}]}\left(p;\rho\right)$ only. Additional simulations could have nevertheless show that distributions for a 125-name portfolio are very closed to their asymptotic counterpart. So, we favor asymptotic distribution to investigate how holders of the different tranche of a CDO are impacted by a change in the default correlation. Holders of the so-called equity tranche are exposed to the first defaults in the portfolios while holders of the last tranche are impacted only if the number of default is significant. To fix idea, let's consider the tranching of the Itraxx contracts for which attachment points are 0%, 3%, 6%, 9%, 12%, 22%. The associated detachment points correspond to the upper limit of the losses covered by the tranche. Figure 5 plots, for each considered tranche, probabilities that losses exceed the detachment point versus the common default correlation ρ.

Insert Figure 5 about here.

Figure 5 exposes how holders of the different tranche are differently impacted by the common default correlation. It can be first observed that holders of the equity tranche (for whom $d_P = 3\bar{\%}$) benefit from any increase of the common default correlation in the sense that the complete loss is less probable. A reason for this is that the underlying references behave more identically as correlation increases. In other words, the common survival correlation increases too. Holders of the other tranches are clearly differently impacted. And among them, investors in the second tranche are rather exposed to the correlation risk.

7 Conclusion

This paper reconsiders the Beta Binomial approach for analyzing homogenous credit portfolios by favoring both the expected default probability and the common default correlation as key parameters. This paper makes standard credit risk indicators functions of the correlation only. So it sheds lights on the model risk associated to that parameter. Analytical expressions have been reported for sensitivity analysis. Simulations show that default correlation is a key parameter to account for. As a final note, it can be pointed out that the idea exposed in the paper is applicable to every mixing distribution admitting a bijective relations with the expected default probability and the common default correlation.

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8 Appendix

Due to the non uniqueness of their representations, various expressions could be derived and reported for the derivatives of the cumulative density function of the beta distribution and its associated inverse function. The below expressions are very appealing for programming on Mathematica - the package used throughout the paper (see the Wolfram's site for more on this).

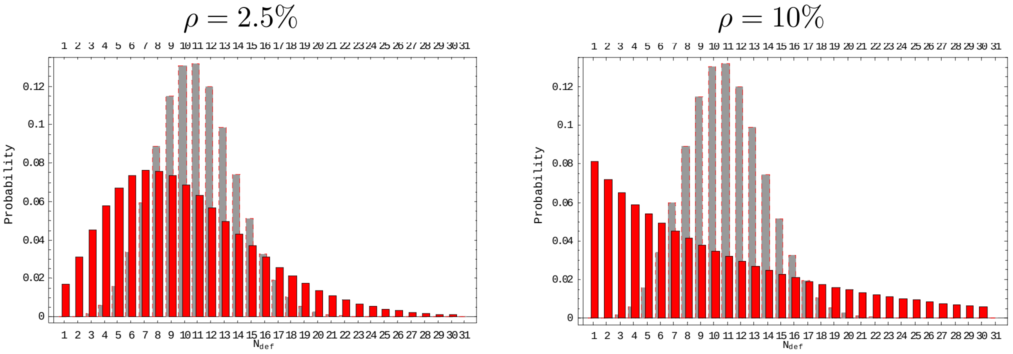
$$\begin{aligned}
\frac{\partial I_x(a,b)}{\partial a} &= \left[\ln(x) - \psi(a) + \psi(a+b)\right] I_x(a,b) \\
&\quad - \frac{\Gamma(a)\Gamma(a+b)}{\Gamma(b)} x^a \, {}_3F_2(a,a,1-b;a+1,a+1;x) \\
\frac{\partial I_x(a,b)}{\partial b} &= -\left[\ln(1-x) - \psi(b) + \psi(a+b)\right] I_{1-x}(b,a) \\
&\quad + \frac{\Gamma(b)\Gamma(a+b)}{\Gamma(a)} (1-x)^b \, {}_3F_2(b,b,1-a;b+1,b+1;1-x) \\
\frac{\partial I_x^{-1}(a,b)}{\partial a} &= \left(1 - I_x^{-1}(a,b)\right)^{1-b} \left[I_x^{-1}(a,b)\right]^{1-a} \times \\
&\quad \Big[-B_{I_x^{-1}(a,b)}(a,b) \left[\ln\left(I_x^{-1}(a,b)\right) - \psi(a) + \psi(a+b)\right] \\
&\quad + \left[I_x^{-1}(a,b)\right]^a \Gamma(a)^2 \; {}_3F_2\left(a,a,1-b;a+1,a+1;I_x^{-1}(a,b)\right)\Big] \\
\frac{\partial I_x^{-1}(a,b)}{\partial b} &= \left(1 - I_x^{-1}(a,b)\right)^{-b} \left(I_x^{-1}(a,b) - 1\right) \left[I_x^{-1}(a,b)\right]^{1-a} \times \\
&\quad \Big[-B_{1-I_x^{-1}(a,b)}(b,a) \left[\ln\left(1 - I_x^{-1}(a,b)\right) - \psi(b) + \psi(a+b)\right] \\
&\quad + \left[1 - I_x^{-1}(a,b)\right]^b \Gamma(b)^2 \; {}_3F_2\left(b,b,1-a;b+1,b+1;1-I_x^{-1}(a,b)\right)\Big]
\end{aligned}$$

where $B_x\left(a,b\right)$ is the incomplete Beta function defined by $B_x\left(a,b\right)=\int_0^x t^{a-1}\left(1-t\right)^{b-1}dt$. ${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;z\right)$ is the regularized Hypergeometric function and ψ is the digamma function. The digamma function is the logarithm derivative of the Euler gamma function: $\psi\left(z\right)=\frac{d}{dz}\ln\Gamma\left(z\right)=\frac{\Gamma'(z)}{\Gamma(z)}$ where $\Gamma\left(z\right)=\int_0^\infty t^{z-1}e^{-t}dt$. The function Γ is a generalization to complex numbers of the factorial function since, for any integer n, $\Gamma\left(n\right)=\left(n-1\right)!$. ${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;z\right)$ is defined by

$${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;z\right)=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_k\left(a_2\right)_k\left(a_3\right)_k}{\left(b_1\right)_k\left(b_2\right)_k}\frac{z^k}{k!}$$

with $\left(a\right)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhammer's symbol. See Abramowitz and Stegun (1972) for more details on these functions.

Figure 1


$\rho = 2.5\%$

$\rho = 10\%$

Figure 1 plots probability density functions of the number of default in a homogenous credit portfolio of $N = 100$ loans or bonds. The shadow probability density function corresponds to the straight binomial model or independent case while the other ones correspond to the reparametrized Beta Mixing Distribution. We set $E[p]$ to 10% and the correlation parameter to either 2.5% or 10%.

Figure 2

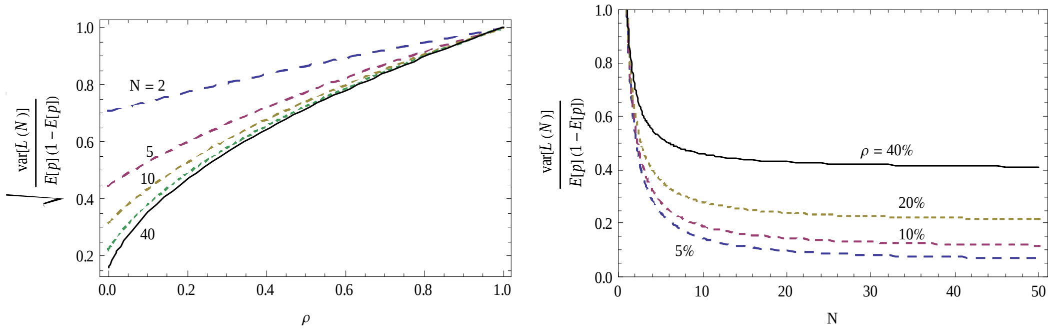


Figure 2 plots the (normalized) variance of the loss rate as a function of N for different values of correlation.

Figure 3

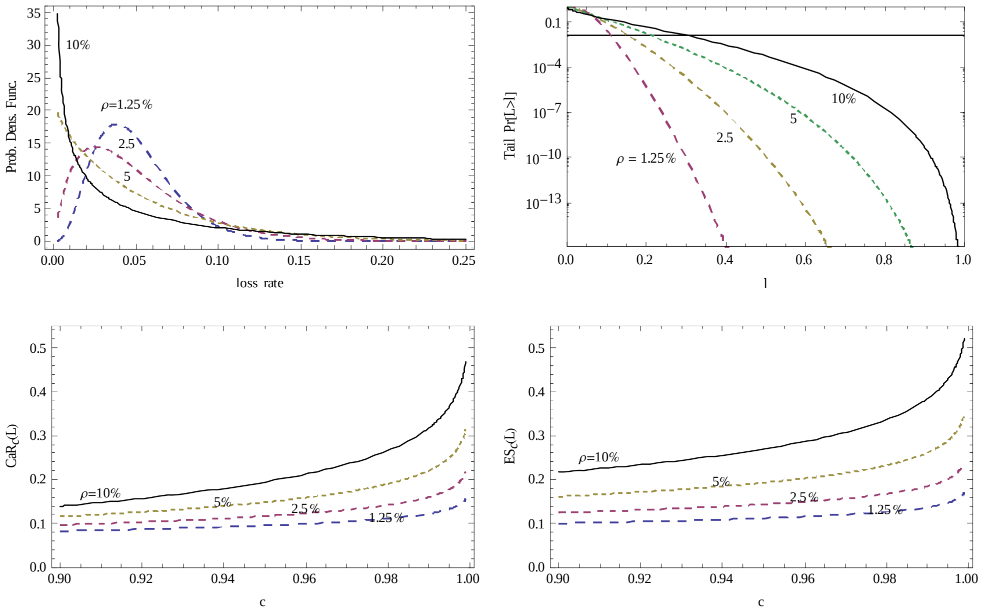
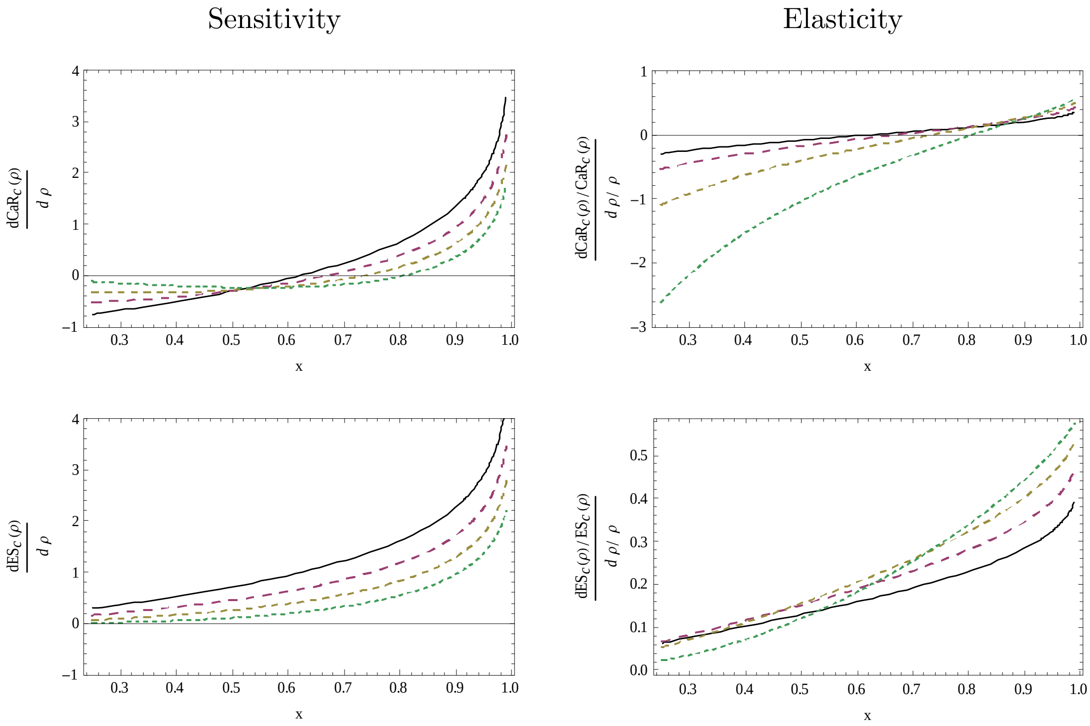


Figure 3 plots the probability density function, the tail function, the Credit at Risk and the Expected Shortfall of the reparametrized Beta Mixing Distribution $\mathcal{I}_{E[\mathbf{p}]}(l;\rho)$ for different values of correlation. The expected default probability of the credit portfolio is equal to 5%. In the upper right graph, the horizontal line stands for the 99-th percentile level.

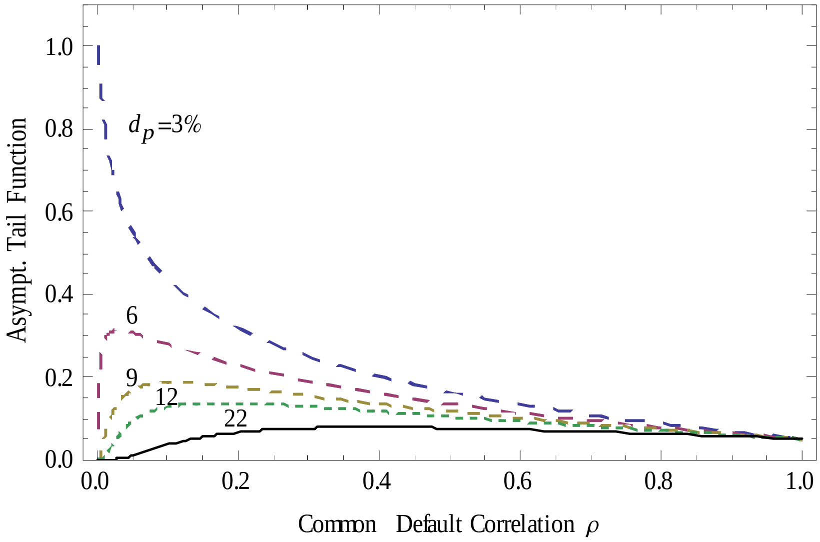
Figure 4


Sensitivity

Elasticity

Figure 4 plots sensitivities and elasticities of the Credit at Risk and the Expected Shortfall to the common default correlation. The expected default probability of the credit portfolio is equal to 5% and the common default correlation parameter is the same as in Figure 3..

Figure 5



d_P stands for detachment point and corresponds to the upper limit of the losses covered by the tranche. The plotted tail function is the probability that the loss exceeds the considered detachment point.

Table 1

	Binomial	Beta Binomial Model (ρ in %)				
$\Pr[N_{Def} = k]$		0*	1.25	2.5	5	10
$k = 0$	59.87	59.87	61.56	63.08	65.75	70.02
$k = 1$	31.51	31.51	28.93	26.71	23.09	17.95
$k = 2$	7.46	7.46	7.59	7.87	7.77	7.08
$k = 3$	1.05	1.05	1.50	1.88	2.44	2.97
$k = 4$	0.10	0.10	0.23	0.39	0.70	1.23
$k = 5$	0.006	0.006	0.027	0.064	0.181	0.486
$k = 6$	0.000	0.000	0.003	0.009	0.041	0.176
$k = 7$	0.000	0.000	0.000	0.001	0.007	0.056
$k = 8$	0.000	0.000	0.000	0.000	0.001	0.015
$k = 9$	0.000	0.000	0.000	0.000	0.000	0.002
$k = 10$	0.000	0.000	0.000	0.000	0.000	0.000
$E[p]$	5	5	5	5	5	5
$\Pr[N_{Def} > 5]$	0.000	0.000	0.003	0.010	0.050	.250

This table provides probabilities of k defaults in a portfolio of 10 credits with . All figures are expressed in %. Bin stands for the simple binomial model described by equation (1). 0* means negligible value.

Table 2

	Default Correlation (in %)		
$\frac{CaR_c(\rho)}{CaR_c(1.25\%)} - 1$	2.5	5	10
$c = 95.00\%$	22.32	55.45	103.85
$c = 97.50\%$	26.64	67.84	131.80
$c = 99.00\%$	31.08	80.52	159.76
$c = 99.90\%$	38.45	100.83	199.98
$c = 99.99\%$	42.82	111.54	215.05
$c = 99.999\%$	45.52	116.90	216.99
$\frac{ES_c(\rho)}{ES_c(1.25\%)} - 1$	2.5	5	10
$c = 95.00\%$	28.21	72.24	140.90
$c = 97.50\%$	31.37	81.21	160.45
$c = 99.00\%$	34.73	90.64	180.07
$c = 99.90\%$	40.56	106.05	207.59
$c = 99.99\%$	44.10	114.13	216.26
$c = 99.999\%$	46.30	117.92	214.87

This table provides relative differences of Credit at Risk and Expected Shortfall for large homogenous credit portfolio. The expected default probability is 5%. All figures are expressed in %.